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ROLE OF KILLING VECTOR FIELDS IN FINSLER SPACE

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ABSTRACT

We consider a Finsler space equipped with a Generalized Conformal β- change of metric and study the Killing vector fields that correspond between the original Finsler space and the Finsler space equipped with Generalized Conformal-change of metric. Also, Killing vector fields of constant length correspond to isometries of constant displacement. Those in turn have been used to study homogeneity of Riemannian and Finsler quotient manifolds.

Keywords: Finsler Space, Riemannian Geometry , Manifolds, Vector Fields.

1.Introduction

In 1976, Hashiguchi **[6]** studied the conformal change of Finsler metrics; namely, $\bar{L} = e^{\sigma(x)}L$

In particular, he also dealt with the special conformal transformation named C-conformal transformation. This change has been studied by Izumi**[3]** and Kropina**[10]** In2008,Abed**[7,8]**introduced the transformation

$$\bar{L} = e^{\sigma(x)}L + \beta$$

thus generalizing the conformal, Randers, and generalized Randers changes. Moreover,

he established the relationships between some important tensors associated with (M,L)

and the corresponding tensors associated with $(M, \bar{L})$.

In this paper, we deal with a general change of Finsler metrics defined by

$$L(x,y) \to \bar{L}(x,y) = f(e^{\sigma(x)}L(x,y), \beta(x,y)) \tag{1.1}$$

where f is a positively homogeneous function of degree one in $\bar{L} = e^{\sigma}L \, and \, \beta$.

This change will be referred to as a generalized βconformal change.

In 1984, Shibata **[1]** studied β-change of Finsler metrics and discussed certain invariant tensors under such a change. Killing equations play important role in the study of a Finsler

space which undergoes a change in the metric. In fact, they give an equivalent characterization for the transformations to preserve distances. In 1979, Singh et al **[9]** studied a Randers space

$$F^n\left(M, L(x,y) = \left(g_{ij}(x)y^i y^j\right)^{\frac{1}{2}} + b_i(x)y^i\right), n \geq 2,$$

which undergoes a change

$$L(x,y) \to L^*(x,y) = L^2(x,y) + (\alpha_i(x)y^i)^2$$

They discussed Killing correspondence of spaces $F^n(M,L) and \, F^{*n}(M,L^*)$.

An isometry ρ of a metric space (M,d) is called Clifford-Wolf (CW) if it moves each

point the same distance, i.e. if the displacement function $\delta(x) = d(x, \rho(x))$ is constant.W. K. Clifford **[11]** described such isometries for the 3-sphere, and G. Vincent **[2]** used the term

Clifford translation for constant displacement isometries of round spheres in his study of spherical space forms S^n/Γ with Γ metabelian. Later J. A. Wolf **([12],[5], [4])** extended the use of the term Clifford translation to the context of metric spaces, especially Riemannian symmetric spaces. There the point is his theorem **[4]** that a complete locally symmetric Riemannian manifold M is homogeneous if and only if, in the universal cover $\tilde{M} \to M = \Gamma \backslash \tilde{M}$, the covering group Γ consist of Clifford translations.

2. In correspondence with $F^n \, and \, \bar{F}^n$ the role of killing vector fields

Let us consider an infinitesimal transformation

$$\acute{x}^i = x^i + \varepsilon v^i(x)$$

Where ε is an infinitesimal constant and $v^i(x)$ is a contravariant vector field.

The vector field $v^i(x)$ is said to be a Killing vector field in F^n if the metric tensor of the Finsler space with respect to the infinitesimal transformation is Lie invariant; that is,

$$\pounds_v g_{ij} = 0, \tag{2.1}$$

With $\pounds_v$ being the operator of Lie differentiation. Equivalently, the vector field $v^i(x)$

is Killing in F^n if

$$v_{i|j} + v_{j|i} + 2C^l_{ij} v_{l|0} = 0 \tag{2.2}$$

Where $v_i = g_{il} v^l$. (2.8)

Now, we prove the following result which gives a necessary and sufficient condition for a Killing vector field in F^n to be Killing in $\bar{F}^n$.

Theorem 1. A Killing vector field $v^i(x)$ in F^n is killing in $\bar{F}^n$ if and only if

$$M_{ij}^l v_{l|0} + C_{rjt} v^t D_j^r + v_r D_{ij}^r + \bar{C}_{ij}^l \left(2C_{r|t} v^t D^r + v_r D_l^r\right) = 0 \qquad (2.3)$$

where $\bar{C}_{ij}^l$ is the associate Cartan tensor of $\bar{F}^n$.

Proof. Assume that $v^i(x)$ is Killing in F^n .Then (2.2) is satisfied. By definition, the h

-covariant derivatives of v_i with respect to $C\bar{\Gamma}$ and C Γ are respectively, given as

$$(a) \quad v_{i||j} = \partial_j v_i - (\dot{\partial}_r v_i)\bar{G}_j^r - v_r \bar{F}_{ij}^r,$$

(2.4)

$$(b) \quad v_{i|j} = \partial_j v_i - (\dot{\partial}_r v_i) G_j^r - v_r F_{ij}^r$$

Where $\partial_j = \partial/\partial x^j$ and "||" denote h-covariant differentiation with respect to $C\bar{\Gamma}$

Equation (2.4)(a), by virtue of (2.4)(b), takes the form

$$v_{i||j} = v_{i|j} - 2C_{rit} v^t D_j^r - v_r D_{ij}^r$$

(2.5)

Now from (2.5), we have

$$v_{i||j} + v_{j||i} + 2\bar{C}_{ij}^l v_{l||0} = v_{i|j} + v_{j|i} + 2C_{ij}^l v_{l|0} - 2C_{rit} v^t D_j^r - 2C_{rjt} v^t D_i^r - 2v_r D_{ij}^r - 2\bar{C}_{ij}^l (2C_{rit} v^t D^r + v_r D_l^r).$$

(2.6)

Using , $\bar{C}_{ij}^l = C_{ij}^l + M_{ij}^l$ in (2.6) and applying (2.2) , we get

$$v_{i||j} + v_{j||i} + 2\bar{C}_{ij}^l v_{l||0} = 2M_{ij}^l v_{l|0} - 2C_{rit} v^t D_j^r - 2C_{rjt} v^t D_i^r - 2v_r D_{ij}^r - 2\bar{C}_{ij}^l (2C_{rit} v^t D^r + v_r D_l^r)$$

(2.7)

Proof is complete with the observation that $v^i(x)$is Killing in $\bar{F}^n$if and only if

$v_{i||j} + v_{j||i} + 2\bar{C}_{ij}^l v_{l||0} = 0$, that is , if and only if (2.3) holds.

If a vector field $v^i(x)$is Killing in F^n and $\bar{F}^n$, then, from Theorem 1, (2.3) holds, which on transvection by y^i yields

$$2C_{rlt} v^t D^r + v_r D_l^r = 0$$

3. Killing vector fields on compact normal homogeneous spaces

Assume $M = G/H$ is a Riemannian normal homogeneous space in which Gis a compact

connected simple Lie group, and H is a closed subgroup with 0<dimH <dimG.

The condition that v defines a non zero CK vector field on M is that $||pr_m(Ad(g)v||$ is a positive constant function of g. For the bi-variant inner product,

$$||v||^2 = ||Ad(g)v||^2 = ||pr_h(Ad(g)v||^2 + ||pr_m(Ad(g)v||^2$$

is a constant function of g$\in G$,, so the same is true for $||pr_h(Ad(g)v||$.Suitably choosing v within its Ad(G)-orbit, we can assume $v \in t$ .Now $\left\|pr_h(\rho(v))\right\|$ *and* $\left\|pr_m(\rho(v))\right\|$ are constant functions of ρ in the Weyl group. Because g is simple and $v \neq 0$, both the functions

$||pr_h(Ad(g)v||$ *and* $||pr_m(Ad(g)v||$ for $g \in G$, are positive constant functions. From the above observations we can state the below Theorem;

Theorem 3.1. *Let G be a compact connected simple Lie group and H a closed subgroup with*

0<dim H <dim G. Fix a normal Riemannian metric on $= G/H$. Suppose that there is a nonzero vector $v \in g$ defining a CK vector field on $M = G/H$ .Then M is a complete locally symmetric Riemannian manifold, and its universal Riemannian cover is an odd dimensional sphere of constant curvature or a Riemannian symmetric space SU(2n)/(Sp(n).

Proposition 3.1. *Let G be a compact connected simple Lie group and H a closed subgroup with 0<dim H <dim G. If $g = a_2$ or g_2 then there is no nonzero $v \in g$ that defines a CK vector field on the Riemannian normal homogeneous space G/H*

Proof. Consider $g = a_2$rst. Assume conversely there is a nonzero CK vector field, defined by the nonzero vector $v \in t$. The subspaces $t \cap h$ *and* $t \cap m$ are a pair of orthogonal

lines in t. Denote all different vectors in the Weyl group orbit of v as $v_1 = v, \ldots\ldots, v_k$,

k=3 or 6, then

$$\sum_{i=1}^{k} v_i = \sum_{i=1}^{k} pr_h(v_i) = \sum_{i=1}^{k} pr_m(v_i) = 0$$

All the vectors $pr_m(v_i)$ have the same nonzero length, which only have two possible

choices in $t \cap m$. So we must have k= 6, and all v_i can be divided into two set, such

that, for example, $pr_m(v_1) = pr_m(v_2) = pr_m(v_3)$ and

$pr_m(v_4) = pr_m(v_5) = pr_m(v_6)$ are opposite to each other. Obviously $v_1 + v_2 + v_3 \neq 0$, so there are two v_i among them, v_1 *and* v_2 for example, such that $v_1 = \rho(v_2)$, in which ρ is the reflection in some root of g. Thus $t \cap h$, containing $v_1 - v_2$, is linearly spanned by a root of g. similar argument can also prove $t \cap m$ is spanned by a root of g. But for a_2, there do not exist a pair of orthogonal roots. This is a contradiction.

The Weyl group of g_2 contains that of a_2 as its subgroup, so the statement for g_2 also follows immediately the above argument.

Conclusion

The main purpose of the present paper is to examine the classical approach to the problem of existence of Killing vector fields and study how they vary from point to point and how they are related to Killing vector fields defined on the whole manifold.

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