



A STUDY OF H-ORTHOGONAL OF TYPE I MATRICES

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ABSTRACT

In this paper, we introduced new types of matrices. We called them horthogonal of type I matrix depend on h-transpose and h-orthogonal. We discussed the properties of these matrices such as, their eigenvalues and determinants. These matrices preserve the length and the inner product.

Keywords: Orthogonal Matrices, H-Orthogonal Matrix, Inner Product, Inverse.

1 Introduction

h-orthogonal of type I is important type of matrices depend on h-transpose . it is a generalization of h-orthogonal matrices and have important properties can apply in important fields such as image processing , data mining and engineering . In this paper, $|||$, det , $<$, $>$, A^h and A^θ means that norm, determinant, inner product, h-transpose and conjugate h-transpose, respectively.

2 Materials and Methods:

Remark 2.1 [2] *If F is a square matrix and l is a nonnegative integer ,*

then

- $(F^T)^l = (F^l)^T$.
- $(F^l)^{-1} = (F^{-1})^l$.

Definition 2.1 [1] *Let $A = [a_{ij}]$ be a $m \times n$ matrix. The **h-transpose** of A is defined by A^h is $n \times m$ matrix such that $A^h = [a^h_{ij}] = [a_{(m+1-j)(n+1-i)}]_{n \times m}$, $i = 1, 2, \dots, m, j = 1, 2, \dots, n$.*

Example 2.2 $A = \begin{bmatrix} 6 & 4 & 7 \\ 4 & -1 & 5 \end{bmatrix}$

$$A^h = \begin{bmatrix} 5 & 7 \\ -1 & 4 \\ 4 & 6 \end{bmatrix}.$$

Theorem 2.3 [1] *If r is a scalar and A and B are matrices of an appropriate size, then:*

- (1) $(A^h)^h = A$.
- (2) $(A^h)^T = (A^T)^h$.
- (3) $(\overline{A})^h = \overline{(A^h)}$.
- (4) $(A + B)^h = A^h + B^h$.
- (5) $(AB)^h = B^h A^h$.
- (6) $(rA)^h = rA^h$
- (7) $(A^h)^{-1} = (A^{-1})^h$, if $det(A) \neq 0$.
- (8) $(A^n)^h = (A^h)^n, n = 2, 3, 4, \dots$

Theorem 2.4 [1] *Let $A \in M_{n \times n}(\mathbb{C})$, then*

- (1) $det(A) = det(A^h) = det(A^T) = det((A^T)^h)$.
- (2) A and A^h have the same eigenvalues.

Remark 2.5 [1] Let $A \in M_{n \times n}(\mathbb{C})$, $A^\theta = (\bar{A})^h = (\overline{A^h})$.

Theorem 2.6 [1] Let $A \in M_{n \times n}(\mathbb{C})$, then

- (1) $(A^\theta)^\theta = A$.
- (2) $(A + B)^\theta = A^\theta + B^\theta$.
- (3) $(AB)^\theta = B^\theta A^\theta$.
- (3) $(kA)^\theta = \bar{k}A^\theta, k \in \mathbb{C}$.
- (5) $(A^n)^\theta = (A^\theta)^n, n = 2, 3, \dots$

Definition 2.2 [1] A matrix $A \in M_{n \times n}(\mathbb{C})$ is called an **h-orthogonal** matrix if $AA^h = I_n$. This means that $A^{-1} = A^h$.

Example 2.7 $A = \begin{bmatrix} 2i & 0 \\ 0 & 1/2i \end{bmatrix}$ is h-orthogonal matrix.

3 Results

Definition 3.1 A matrix $A \in M_{n \times n}(\mathbb{C})$ is called **h-orthogonal of type I** if $A^k(A^h)^k = I_n$, for some $k \in \mathbb{N}$.

Example 3.1 $\bullet A = \begin{bmatrix} \sqrt{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\sqrt{i} \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

A and B are h-orthogonal of type I.

$\bullet C = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is not h-orthogonal of type I matrix.

3.1 The Index of h-Orthogonal of Type I

Definition 3.2 Let A be an h-orthogonal of type I matrix. The smallest positive integer k with $A^k(A^h)^k = I_n$ is called the **index** of A. In such case, we say that A is an h-orthogonal of type I of **period k**. Or **k-period** of h-orthogonal of type I matrix.

Example 3.2 Let

$$A = \begin{bmatrix} \sqrt{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\sqrt{i} \end{bmatrix}$$

$$\text{Then } A^h = \begin{bmatrix} \sqrt{i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\sqrt{i} \end{bmatrix}$$

$$k = 1, AA^h = \begin{bmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$$k = 2, A^2(A^h)^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$k = 3, A^3(A^h)^3 = \begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix}$$

$$k = 4, A^4(A^h)^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$k = 5, A^5(A^h)^5 = \begin{bmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$$k = 6, A^6(A^h)^6 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$k = 7, A^7(A^h)^7 = \begin{bmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix}$$

$$k = 8, A^8(A^h)^8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The index of A is 4. $A^m(A^h)^m = I_n, m = 4, 8, 12, \dots$, and A is of period 4.

Theorem 3.3 If A is an h -orthogonal

of type I matrix with index k , then $\det(A^k) = \pm 1$.

Proof:

Let A be an h -orthogonal of type I matrix with index k , then $A^k(A^h)^k = I_n$.

$$\Rightarrow \det(A^k(A^h)^k) = \det(I_n)$$

$$\Rightarrow \det(A^k(A^h)^k) = 1$$

$$\Rightarrow \det(A^k) \det(A^h)^k = 1$$

$$\Rightarrow \det(A^k) \det(A^k) = 1 \text{ [2.4]}$$

$$\Rightarrow \det(A^{2k}) = 1$$

$$\Rightarrow \det(A^k)^2 = 1$$

$$\Rightarrow \det(A^k) = \pm 1$$

□

3.2 Inverse of h -Orthogonal of Type I

Theorem 3.4 If A is h -orthogonal of type I matrix with index k , then it is invertible with

$$A^{-1} = A^{k-1}(A^h)^k$$

Proof:

suppose that A is h -orthogonal of type I matrix with index k .

Then $A^k(A^h)^k = I_n$.

this equivalent to

$$(A^k)^{-1} = (A^h)^k$$

$$\Rightarrow (A^{-1})^k = (A^h)^k$$

Multiplying by $A^{k-1} \Rightarrow A^{-1}(A^{-1})^{k-1} = (A^h)^k$

Hence

$$A^{-1} = A^{k-1}(A^h)^k$$

□

Theorem 3.5 Let $A \in M_{n \times n}(\mathbb{C})$, then the following statements are equivalent:

- (1) A is an h -orthogonal of type I matrix.
- (2) A^{-1} is an h -orthogonal of type I matrix.
- (3) A^T is an h -orthogonal of type I matrix.
- (4) A^* is an h -orthogonal of type I matrix.
- (5) \bar{A} is an h -orthogonal of type I matrix.
- (6) A^h is an h -orthogonal of type I matrix.
- (7) A^θ is an h -orthogonal of type I matrix.

Proof:

(1 \Rightarrow 2)

suppose that A is an h -orthogonal of type I matrix.

so $A^k(A^h)^k = I_n$, for some $k \in \mathbb{N}$

$$\Rightarrow (A^k(A^h)^k)^{-1} = (I_n)^{-1}$$

$$((A^{-1})^h)^k(A^{-1})^k = I_n$$

Hence A^{-1} is h -orthogonal of type I matrix.

(2 \Rightarrow 3)

suppose that A^{-1} is an h -orthogonal of type I matrix

so $(A^{-1})^k((A^{-1})^h)^k = I_n$, for some $k \in \mathbb{N}$

$$\Rightarrow (A^k(A^h)^k)^{-1} = I_n$$

$$\Rightarrow ((A^h)^k A^k) = I_n$$

$$\Rightarrow ((A^h)^k A^k)^T = I_n^T$$

$$\Rightarrow (A^T)^k((A^h)^k)^T = I_n$$

$$\Rightarrow (A^T)^k((A^T)^h)^k = I_n$$

Hence A^T is an h -orthogonal of type I matrix.

(3 \Rightarrow 4)

suppose that A^T is an h -orthogonal of type I matrix.

so $(A^T)^k((A^T)^h)^k = I_n$, for some $k \in \mathbb{N}$

$$\Rightarrow \overline{(A^T)^k((A^T)^h)^k} = \overline{I_n}$$

$$\Rightarrow \overline{(A^T)^k(A^T)^h)^k} = I_n$$

$$\Rightarrow (A^*)^k((A^*)^h)^k = I_n.$$

Hence A^* is h -orthogonal of type I matrix.

(4 \Rightarrow 5)

suppose that A^* is an h -orthogonal of type I matrix.

so $(A^*)^k((A^*)^h)^k = I_n$, for some $k \in \mathbb{N}$

$$\Rightarrow ((A^*)^k((A^*)^h)^k)^T = I_n^T$$

$$\Rightarrow ((A^*)^h)^k((A^*)^k)^T = I_n$$

$$\Rightarrow ((\overline{A})^h)^k(\overline{A})^k = I_n$$

Hence \overline{A} is h -orthogonal of type I matrix.

(5 \Rightarrow 6)

suppose that \overline{A} is an h -orthogonal of type I matrix.

so $(\overline{A})^k((\overline{A})^h)^k = I_n$, for some $k \in \mathbb{N}$

$$\Rightarrow \overline{((\overline{A})^k(\overline{A})^h)^k} = I_n$$

$$\Rightarrow (\overline{A})^k((\overline{A})^h)^k = I_n$$

$$\Rightarrow A^k(A^h)^k = I_n$$

$$\Rightarrow ((A^h)^h)^k(A^h)^k = I_n$$

Hence A^h is h -orthogonal of type I matrix.

(6 \Rightarrow 7)

suppose that A^h is an h -orthogonal of type I matrix.

so $(A^h)^k((A^h)^h)^k = I_n$, for some $k \in \mathbb{N}$

$$\Rightarrow (A^h)^k A^k = I_n$$

$$\Rightarrow ((A^h)^k A^k)^\theta = I_n^\theta$$

$$\Rightarrow (A^\theta)^k((A^h)^k)^\theta = I_n$$

$$\Rightarrow (A^\theta)^k((A^\theta)^h)^k = I_n$$

Hence A^θ is h -orthogonal of type I matrix.

(7 \Rightarrow 1)

suppose that A^θ is an h -orthogonal of type I matrix.

so $(A^\theta)^k((A^\theta)^h)^k = I_n$, for some $k \in \mathbb{N}$

$$\begin{aligned} \Rightarrow & ((A^\theta)^k((A^\theta)^h)^k)^\theta = I_n^\theta \\ \Rightarrow & ((A^\theta)^h)^k((A^\theta)^k)^\theta = I_n \\ \Rightarrow & (A)^k(A^h)^k = I_n \end{aligned}$$

Hence A is h -orthogonal of type I matrix. \square

Theorem 3.6 A matrix $A \in M_{n \times n}(\mathbb{C})$ is h -orthogonal of type I if and only if A^m is h -orthogonal of type I matrix for each $m \in \mathbb{N} \setminus \{1\}$.

Proof:

suppose that A is an h -orthogonal of type I matrix.

so

$$\begin{aligned} A^k(A^h)^k &= I_n \text{ for some } k \in \mathbb{N} \\ \Rightarrow (A^k)^m((A^h)^k)^m &= (I_n)^m \\ \Rightarrow (A^m)^k((A^m)^h)^k &= I_n \end{aligned}$$

Hence A^m is h -orthogonal of type I matrix.

Now, suppose that A^m is h -orthogonal of type I matrix of index k for each $m \in \mathbb{N} \setminus \{1\}$; specially, each of A^2 and A^5 is an h -orthogonal of type I matrix of index k .

So.

$$\Rightarrow I_n = A^k(A^4)^k((A^4)^h)^k(A^h)^k$$

So $A^k(A^h)^k = I_n$ for A^2 is h -orthogonal of type I

Hence A is h -orthogonal of type I matrix of index k . \square

Theorem 3.7 If A is an h -orthogonal of type I matrix of index k , then each of $A^T, A^{-1}, \bar{A}, A^h, A^\theta$ and A^* is h -orthogonal of type I of index k .

Proof: (1) Suppose that A is an h -orthogonal of type I matrix of index k , then A^h is h -orthogonal of type I matrix with $\text{ind}(A^h) \leq k$ [3.5]

Now, suppose that

$$\text{ind}(A^h) = k - m, \quad 1 \leq m < k$$

$$\text{So } (A^h)^{k-m}A^{k-m} = I$$

Then $\text{ind}(A) = k - m$ which is a contradiction

Hence $\text{ind}(A^h) = k$.

(2) By [3.5] A^θ is an h -orthogonal of type I matrix with $\text{ind}(A^\theta) \leq k$.

suppose that $\text{ind}(A^\theta) = k - m, 1 \leq m < k$.

Then

$$\begin{aligned} (A^\theta)^{k-m}((A^\theta)^h)^{k-m} &= I \\ \Rightarrow (A^\theta)^{k-m}(\bar{A})^{k-m} &= I \\ \Rightarrow ((\bar{A})^h)^{k-m}(\bar{A})^{k-m} &= I. \end{aligned}$$

Taking the conjugate of both side:

$$\Rightarrow (A^h)^{k-m}A^{k-m} = I$$

Thus $\text{ind}(A) = k - m$ which is a contradiction

Hence $\text{ind}(A^\theta) = k$.

Similarly with respect to A^{-1}, A^*, A^T and \bar{A} . \square

Theorem 3.8 If A is h -orthogonal of type I , then $(A^h)^T$ is h -orthogonal of type I matrix.

Proof:

suppose that A is h -orthogonal of type I matrix, then A^θ is h -orthogonal of type I [Theorem 3.5]

So, $(A^\theta)^k((A^\theta)^h)^k = I_n$ for some $k \in \mathbb{N}$

$$\begin{aligned} \Rightarrow & ((A^\theta)^k((A^\theta)^h)^k)^* = (I_n)^* \\ \Rightarrow & ((A^\theta)^h)^k((A^\theta)^k)^* = I_n \\ \Rightarrow & ((A^\theta)^*)^h)^k((A^\theta)^*)^k = I_n \\ \Rightarrow & ((A^h)^T)^h)^k((A^h)^T)^k = I_n \end{aligned}$$

Hence $(A^h)^T$ is h -orthogonal of type I matrix. \square

Theorem 3.9 If $(A^h)^T$ is h -orthogonal of type I matrix, then $(A^\theta)^T$ is h -orthogonal of type I matrix.

Proof: suppose that $(A^h)^T$ is h -orthogonal of type I matrix, then

$((A^h)^T)^k((A^h)^T)^h)^k = I_n$, for some $k \in \mathbb{N}$

$$\begin{aligned} \Rightarrow & \overline{((A^h)^T)^h)^k((A^h)^T)^k} = \overline{I_n} \\ \Rightarrow & \overline{(((A^h)^T)^h)^k}(\overline{((A^h)^T)^k}) = I_n \\ \Rightarrow & \overline{(((A^\theta)^T)^h)^k}(\overline{(A^\theta)^T})^k = I_n \end{aligned}$$

Hence $(A^\theta)^T$ is h -orthogonal of type I matrix. \square

Theorem 3.10 If λ is an eigenvalue of an h -orthogonal of type I matrix A with index k , then λ is of modulus 1

Proof: Let A be an h -orthogonal of type I matrix with index k , then

$$A^k(A^h)^k = I_n$$

$$\Rightarrow (A^k)^{-1} = (A^h)^k = (A^k)^h$$

Since λ is an eigenvalue of A , then λ^k is an eigenvalue of A^k and $1/\lambda^k$ is an eigenvalue of $(A^k)^{-1}$ and $(\lambda^k)^h$ is eigenvalue of $(A^k)^h$ [??]

Since A and A^h have the same eigenvalues [2.4], then

$$\lambda^k = 1/\lambda^k$$

$$\text{So } (\lambda^k)^2 = 1$$

$$\text{Then } |(\lambda^k)^2| = 1$$

$$|\lambda^k| = \underbrace{|\lambda||\lambda|\dots|\lambda|}_{(2k\text{-times})} = 1$$

Since $|\lambda| > 0$ and real number, then we must have

$$|\lambda| = 1$$

\square

Theorem 3.11 If $A_{n \times n}$ is an h -orthogonal of type I , then it preserves the inner product in the eigenvectors subspace.

Proof: Let λ be an eigenvalue of A and x and y be an eigenvectors corresponding to λ .

then

$$Ax = \lambda x$$

and

$$Ay = \lambda y$$

$$\begin{aligned} \text{So } \langle Ax, Ay \rangle &= \langle \lambda x, \lambda y \rangle \\ &= \lambda \bar{\lambda} \langle x, y \rangle \\ &= |\lambda|^2 \langle x, y \rangle \\ &= \langle x, y \rangle \text{ [Theorem 3.10]} \end{aligned}$$

□ If $A_{n \times n}$ is an h -orthogonal

of type I , then it preserves the length in the eigenvectors subspace. **Proof:** Let λ be an eigenvalue of A and x be an eigenvector corresponding to λ .

$$\langle Ax, Ax \rangle = \langle x, x \rangle \text{ [Theorem 3.11]}$$

$$\|Ax\|^2 = \|x\|^2$$

$$\|Ax\| = \|x\|.$$

□

Theorem 3.12 If A and B are two commute h -orthogonal of type I matrices of index k , then AB are h -orthogonal of type I matrix.

Proof: since A and B are h -orthogonal of type I matrices, then $A^k(A^h)^k = I_n$ and $B^k(B^h)^k = I_n$

$$\begin{aligned} \Rightarrow (AB)^k((AB)^h)^k &= (AB)^k(B^hA^h)^k \\ &= A^k B^k (B^h)^k (A^h)^k \\ &= A^k (A^h)^k \\ &= I_n \end{aligned}$$

Hence AB is h -orthogonal of type I matrix.

If A and B have different indices, then [3.12] satisfies :

Let A and B be h -orthogonal of type I matrices with indices k_1 and k_2 respectively, then $(AB)^k((AB)^h)^k = I_n$, where k is least common multiple of k_1 and k_2 .

□

Remark 3.13 The condition of commute is necessary to make [3.12] true.

Example 3.14

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix}, AB = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \neq BA$$

A and B are h -orthogonal of type I

$$(AB)(AB)^h = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$(AB)^2((AB)^h)^2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$$

$$(AB)^3((AB)^h)^3 = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

$$(AB)^4((AB)^h)^4 = \begin{bmatrix} 1 & 0 \\ -8 & 1 \end{bmatrix}$$

$$(AB)^k((AB)^h)^k = \begin{bmatrix} 1 & 0 \\ -(2k) & 1 \end{bmatrix} \neq I_2, \text{ for every } k$$

Remark 3.15 Every h -orthogonal matrix is h -orthogonal of type I matrix, but the converse is not true.

Remark 3.16 If A and B are two h -orthogonal of type I matrices with index k , then $A+B$ need not be h -orthogonal of type I

Example 3.17

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \text{ and } B = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix}$$

are h -orthogonal of type I matrix of index $k_1 = 1, k_2 = 2$

$$A + B = \begin{bmatrix} 2i & 0 \\ i & -2i \end{bmatrix}$$

$$(A + B)^h = \begin{bmatrix} -2i & 0 \\ i & 2i \end{bmatrix}$$

$$(A + B)(A + B)^h = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix}$$

$$(A + B)^2((A + B)^h)^2 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

$$(A + B)^3((A + B)^h)^3 = \begin{bmatrix} 64 & 0 \\ 64 & 64 \end{bmatrix}$$

$$(A + B)^4((A + B)^h)^4 = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$$

$$(A + B)^k((A + B)^h)^k = \begin{bmatrix} 2^{2k} & 0 \\ 2^{2k} & 2^{2k} \end{bmatrix}$$

for k is odd

$$(A + B)^k((A + B)^h)^k = \begin{bmatrix} 2^{2k} & 0 \\ 0 & 2^{2k} \end{bmatrix}$$

for k is even

Then $(A + B)^k((A + B)^h)^k \neq I_2$ for any $k \in \mathbb{N}$.

So $A + B$ is not h -orthogonal of type I matrix.

Remark 3.18 If A is h -orthogonal of type I matrix and c is a scalar, then cA need not be h -orthogonal of type I.

Example 3.19 $A = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix}$ is h -orthogonal of type I matrix with index $k = 2$, let $c = 2$

$$cA = \begin{bmatrix} 2i & 0 \\ 2i & -2i \end{bmatrix}$$

$$(cA)^h = \begin{bmatrix} -2i & 0 \\ 2i & 2i \end{bmatrix}$$

$$(cA)(cA)^h = \begin{bmatrix} 4 & 0 \\ 8 & 4 \end{bmatrix}$$

$$(cA)^2((cA)^h)^2 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$$

$$(cA)^3((cA)^h)^3 = \begin{bmatrix} 64 & 0 \\ 128 & 64 \end{bmatrix}$$

$$(cA)^4((cA)^h)^4 = \begin{bmatrix} 256 & 0 \\ 0 & 256 \end{bmatrix}$$

$$(cA)^k((cA)^h)^k = \begin{bmatrix} 4^k & 0 \\ 2(4^k) & 4^k \end{bmatrix}, \text{ for } k \text{ is odd}$$

$$(cA)^k((cA)^h)^k = \begin{bmatrix} 4^k & 0 \\ 0 & 4^k \end{bmatrix}, \text{ for } k \text{ is even}$$

Then $(cA)^k((cA)^h)^k \neq I_2 \forall k \in \mathbb{N}$.

So $2A$ need not be h -orthogonal of type I matrix.

If A is a normal matrix, then A^h is normal. **Proof:** Suppose that A is a normal matrix, then $A^*A = AA^*$.

$$\Rightarrow A^h(A^h)^* = A^h(A^*)^h$$

$$= (A^*A)^h$$

$$= (AA^*)^h$$

$$= (A^*)^h A^h = (A^h)^* A^h$$

Hence A^h is normal □ If

A is a normal matrix, then A^θ is normal. **Proof:** Suppose that A is a normal matrix, then $A^*A = AA^*$.

$$\begin{aligned} \Rightarrow A^\theta(A^\theta)^* &= A^\theta(A^*)^\theta \\ &= (A^*A)^\theta \\ &= (AA^*)^\theta \\ &= (A^*)^\theta A^\theta = (A^\theta)^*A^\theta \end{aligned}$$

Hence A^θ is normal □

Theorem 3.20 Let A is h -orthogonal of type I :

1- If $A\bar{A} = \bar{A}A$, then $A\bar{A}$ and $\bar{A}A$ are h -orthogonal of type I.

2- If A normal then AA^* and A^*A are h -orthogonal of type I.

3- If $AA^T = A^T A$, then AA^T and $A^T A$ are h -orthogonal of type I.

4- If $AA^\theta = A^\theta A$, then AA^θ and $A^\theta A$ are h -orthogonal of type I.

Proof: (1) Suppose that A is h -orthogonal of type I matrix with index k and $A\bar{A} = \bar{A}A$

$$\begin{aligned} (A\bar{A})^k((A\bar{A})^h)^k &= (A\bar{A})^k((\bar{A})^h(A)^h)^k \\ &= A^k(\bar{A})^k((\bar{A})^h)^k(A^h)^k \\ &= A^k I_n (A^h)^k \\ &= I_n \end{aligned}$$

Hence $A\bar{A}$ is h -orthogonal of type I matrix.

similarly that $\bar{A}A$.

(2) Suppose that A is normal h -orthogonal of type I matrix with index k By [Lemma3], A^h is normal,

$$\text{So } A^h(A^h)^* = (A^h)^*A^h$$

$$\begin{aligned} (AA^*)^k((AA^*)^h)^k &= (AA^*)^k((A^*)^h A^h)^k \\ &= A^k(A^*)^k((A^*)^h)^k(A^h)^k \\ &= A^k I_n (A^h)^k \\ &= A^k(A^h)^k = I_n \\ \Rightarrow (AA^*)^k((AA^*)^h)^k &= I_n \end{aligned}$$

Hence AA^* is h -orthogonal of type I matrix.

similarly that A^*A .

(3) Suppose that A is h -orthogonal of type I matrix with index k and $AA^T = A^T A$.

$$\begin{aligned} (A^T A)^k((A^T A)^h)^k &= (A^T A)^k((A^h(A^T)^h)^k \\ &= (A^T)^k A^k (A^h)^k ((A^T)^h)^k \\ &= (A^T)^k I_n ((A^T)^h)^k \\ &= (A^T)^k ((A^T)^h)^k = I_n \\ \Rightarrow (A^T A)^k((A^T A)^h)^k &= I_n \end{aligned}$$

Hence $A^T A$ is h -orthogonal of type I matrix.

similarly that AA^T .

(4) Suppose that A is h -orthogonal of type I matrix with index k and $AA^\theta = A^\theta A$.

$$\begin{aligned} (AA^\theta)^k((AA^\theta)^h)^k &= (AA^\theta)^k((A^\theta)^h A^h)^k \\ &= A^k(A^\theta)^k((A^\theta)^h)^k(A^h)^k \\ &= A^k I_n (A^h)^k = I_n \\ \Rightarrow (AA^\theta)^k((AA^\theta)^h)^k &= I_n \end{aligned}$$

Hence AA^θ is h -orthogonal of type I matrix.

similarly that $A^\theta A$ □

Theorem 3.21 If A be h -orthogonal of type I matrix with index k and $\overline{AA^h} = A^h\overline{A}$, then $\overline{AA^h}$ is h -orthogonal of type I matrix.

Proof: Suppose that A is h -orthogonal of type I matrix with index k

$$\begin{aligned} (\overline{AA^h})^k((\overline{AA^h})^h)^k &= (\overline{AA^h})^k(A(\overline{A})^h)^k \\ &= (\overline{A})^k(A^h)^kA^k((\overline{A})^h)^k \\ &= (\overline{A})^kI_n((\overline{A})^h)^k \\ &= (\overline{A})^k((\overline{A})^h)^k = I_n \\ \Rightarrow (\overline{AA^h})^k((\overline{AA^h})^h)^k &= I_n \end{aligned}$$

Hence $\overline{AA^h}$ is h -orthogonal of type I matrix □

Theorem 3.22 If A is h -orthogonal of type I matrix with index k and $A^*A^h = A^hA^*$, then A^*A^h is h -orthogonal of type I matrix.

Proof: suppose that A is h -orthogonal of type I matrix with index k

$$\begin{aligned} (A^*A^h)^k((A^*A^h)^h)^k &= (A^*A^h)^k(A(A^*)^h)^k \\ &= (A^*)^k(A^h)^kA^k((A^*)^h)^k \\ &= (A^*)^kI_n((A^*)^h)^k \\ &= I_n \\ \Rightarrow (A^*A^h)^k((A^*A^h)^h)^k &= I_n \end{aligned}$$

Hence A^*A^h is h -orthogonal of type I matrix. □

Theorem 3.23 If AA^T is h -orthogonal of type I matrix with index k and A is symmetric, then A is h -orthogonal of type I matrix.

Proof:

Suppose that AA^T is h -orthogonal of type I matrix with index k .

To prove AA^T is h -orthogonal of type I matrix,

$$\begin{aligned} (AA^T)^k((AA^T)^h)^k &= I_n \\ \Rightarrow (AA)^k((AA)^h)^k &= I_n \\ \Rightarrow (A^2)^k((A^2)^h)^k &= I_n \end{aligned}$$

for A is symmetric $\Rightarrow A^{2k}(A^h)^{2k} = I_n$

Hence A is h -orthogonal of type I matrix. □

Theorem 3.24 If A is h -orthogonal of type I matrix with index k and $A^hA^* = A^*A^h$, then $(A^hA^*)^k$ is unitary.

Proof:

Suppose that A is h -orthogonal of type I matrix with index k , to prove $(A^hA^*)^k$ is unitary:

$$\begin{aligned} ((A^hA^*)^k)^*(A^hA^*)^k &= ((A^hA^*)^*)^k(A^hA^*)^k \\ &= (A(A^h)^*)^k(A^*A^h)^k \\ &= A^k((A^h)^*)^k(A^*)^k(A^h)^k \\ &= A^k((A^*)^hA^*)^k(A^h)^k \\ &= A^kI_n(A^h)^k \\ &= I_n \end{aligned}$$

Hence $(A^hA^*)^k$ is unitary matrix. □

Theorem 3.25 If AA^* is h -orthogonal of type I matrix with index k and A is Hermitian matrix, then A is h -orthogonal of type I matrix.

Proof:

Suppose that AA^* is h -orthogonal of type I matrix with index k

$$\text{Then } (AA^*)^k((AA^*)^h)^k = I_n$$

$$\Rightarrow (AA)^k((AA)^h)^k = I_n$$

$$\Rightarrow (A^2)^k((A^2)^h)^k = I_n$$

$$\Rightarrow A^{2k}(A^h)^{2k} = I_n$$

Hence A is h -orthogonal of type I matrix □

Theorem 3.26 If A is h -orthogonal of type I matrix with index k and $A^T A^h = A^h A^T$, then $(A^h A^T)^k$ is orthogonal.

Proof: Suppose that A is h -orthogonal of type I matrix with index k , to prove $(A^h A^T)^k$ is orthogonal:

$$((A^h A^T)^k)^T (A^h A^T)^k = ((A^h A^T)^T)^k (A^h A^T)^k$$

$$= (A(A^h)^T)^k (A^T A^h)^k$$

$$= A^k ((A^T)^h)^k (A^T)^k (A^h)^k$$

$$= A^k I_n (A^h)^k$$

$$= I_n$$

Hence $(A^T A^h)^k$ is orthogonal matrix.

□

4. Conclusion

h -orthogonal of type I matrix is a generalization of the h -orthogonal matrix. These matrices are invertible and its eigenvalues are of modulus 1. Every h -orthogonal of type I matrix A of index k induces a set of h -orthogonal matrices.

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