Research Paper
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CONVOLUTION POLYNOMIAL DIVISION TEMPLATE

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ABSTRACT

The division of a pair of giving polynomials to find its quotient and remainder is derived by applying the convolution matrix. The process of matrix formulation with compact template is simple, efficient and direct, comparing to the familiar classical longhand division and synthetic polynomial division.

Typical numerical examples are provided to show the merit of the approaches.

Keywords: Polynomial division; Longhand division; Synthetic division; Convolution matrix; Matrix formulation.

Introduction and Formulation

The division of two univariate polynomials to find the quotient and remainder is expressed as

$$\frac{b(x)}{a(x)} = q(x) + \frac{r(x)}{a(x)}$$

or

$$b(x) = a(x)\,q(x) + r(x)$$

where $b(x)$ and $a(x)$ are the given dividend and divisor of degrees n and m, n ³ m , and $q(x)$ and $r(x)$ are the resulted quotient and remainder of degrees $n - m$ and $m - 1$ or less, respectively,

$$b(x) = b_0x^n + b_1x^{n-1} + \cdots\cdots + b_{n-1}x + b_n$$
$$a(x) = a_0x^m + a_1x^{m-1} + \cdots\cdots + a_{m-1}x + a_m$$

$$q(x) = q_0x^{n-m} + q_1x^{n-m-1} + \cdots\cdots + q_{n-m-1}x + q_{n-m}$$
$$r(x) = r_0x^{m-1} + r_1x^{m-2} + \cdots\cdots + r_{m-2}x + r_{m-1}$$

Then the coefficients of x^ℓ in both side of polynomial division equation after substituting of the expansion forms of $b(x)$, $a(x)$ and $q(x)$, $r(x)$ will give the following relation:

$$b_\ell = a_\ell q_0 + a_{\ell-1}q_1 + \cdots\cdots + a_1 q_{\ell-1} + a_0 q_\ell + r_{\ell-(n-m+1)}, \qquad \ell = 0, 1, \cdots\cdots, n$$

where it is understood that

$$b_\ell = 0,\ \ell > n, \quad a_\ell = 0,\ \ell > m, \quad \text{and} \quad q_\ell = 0,\ \ell > n-m, \quad r_\ell = 0,\ \ell < 0.$$

The polynomial division is then written in the convolution matrix form [1,2] as:

$$\begin{bmatrix} b_0x^n \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_0x^m & & & & 0 & & \\ \vdots & \ddots & & & & \ddots & \\ \vdots & & \ddots & & & & \ddots \\ \vdots & & & a_0x^m & & & 0 \\ \vdots & & & \vdots & 1 & & \\ a_m & & & \vdots & & \ddots & \\ & \ddots & & \vdots & & & \ddots \\ & & \ddots & \vdots & & & \ddots \\ & & & a_m & & & 1 \end{bmatrix} \begin{bmatrix} q_0x^{n-m} \\ \vdots \\ \vdots \\ q_{n-m} \\ r_0x^{m-1} \\ \vdots \\ \vdots \\ \vdots \\ r_{m-1} \end{bmatrix}$$

In practical calculation, we can omit the x factor in this linear algebraic equation. Also if its square matrix is found to be non-singular, the desired coefficients of q(x) and r(x) can thus be uniquely obtained.

This algebraic matrix equation may also be further separated into two sets of equations for computing our desired entries:

$$\begin{bmatrix} b_0 \\ \vdots \\ \vdots \\ b_{n-m} \end{bmatrix} = \begin{bmatrix} a_0 & & & \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ a_{n-m} & \cdots & \cdots & a_0 \end{bmatrix} \begin{bmatrix} q_0 \\ \vdots \\ \vdots \\ q_{n-m} \end{bmatrix}$$

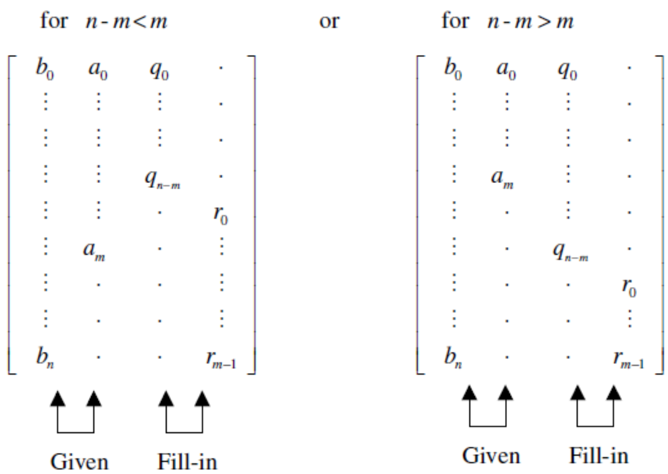
and

$$\begin{bmatrix} b_{n-m+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{n-m+1} & \cdots & \cdots & a_1 \\ \vdots & & & \vdots \\ a_m & & & \vdots \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & & a_m \end{bmatrix} \begin{bmatrix} q_0 \\ \vdots \\ \vdots \\ q_{n-m} \end{bmatrix} + \begin{bmatrix} r_0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ r_{m-1} \end{bmatrix}$$

Here, from the given entries $\mathbf{b} = [b_0, b_1, \cdots, b_n]^{\mathrm{T}}$ and $\mathbf{a} = [a_0, a_1, \cdots, a_m]^{\mathrm{T}}$ the desired entrie $\mathbf{q} = [q_0, q_1, \cdots, q_{n-m}]^T$ must be computed consecutively in the entry order, and the

entries $\mathbf{r} = [r_0, r_1, \cdots, r_{m-1}]^T$ may then be found accordingly.

Applying the relationship among all of these entries, the polynomial division manipulation may further be caste into either of the following templates:



The desired entries of quotient and reminder can then be directed determined without writing down any intermediate data as in the familiar longhand polynomial division and synthetic polynomial division.

It follows that our desired coefficients may also be recursively computed by the following formulas:

$$q_k = \left(b_k - \sum_{\ell=\max(0,k-m)}^{k-1} a_{k-\ell}\, q_\ell \right) \Big/ a_0 \,, \qquad k = 0, \cdots, n-m$$

$$r_{k-(n-m+1)} = \left(b_k - \sum_{\ell=\max(0,k-m)}^{n-m} a_{k-\ell}\, q_\ell \right), \qquad k = n-m+1, \cdots, n$$

A simple computer routine in MATLAB is presented. Inputs b and a, and outputs q and r are the coefficient vectors of given dividend $b(x)$ and divisor $a(x)$, and resulted quotient $q(x)$ and reminder $r(x)$, respectively.

```
function [q,r] = poly_div(b,a)
%   Polynomial division by template.
%   Similar to MATLAB built-in function: 'deconv.m'.
%   F C Chang       08/28/2017

       n = length(b)-1;  m = length(a)-1;
    if m > n,   q = 0;  r = b;   return;  end;
       a = [a,zeros(1,n-m)];     q = 0;
  for  k = 1:n+1,
    if k < n-m+2,
       q(k) = (b(k)-[q(1:k-1)]*[a(k:-1:2)].')/a(1); r = 0;
    else
       r(k-(n-m+1)) = b(k)-[q(1:n-m+1)]*[a(k:-1:k-n+m)].';
    end
  end;
```

Typical Example with Remarks

For given

$$b(x) = 4x^8 + 5x^7 - x^6 + 7x^5 - 6x^4 + x^3 + 2x^2 - 3x + 7$$

$$a(x) = 3x^5 + x^4 - 7x^3 + 5x^2 - 4x + 2$$

in

$$\frac{b(x)}{a(x)} = q(x) + \frac{r(x)}{a(x)}$$

we shall find the desired results as

$$q(x) = \frac{4}{3}x^3 + \frac{11}{9}x^2 + \frac{64}{27}x + \frac{176}{81}$$

$$r(x) = \frac{619}{81}x^4 + \frac{533}{81}x^3 - \frac{148}{81}x^2 + \frac{77}{81}x + \frac{215}{81}$$

by applying either one of the following approaches:

(1) Longhand polynomial division

$$\begin{array}{r|rrrrrrrrr}
 & +\frac{4}{3} & +\frac{11}{9} & +\frac{64}{27} & +\frac{176}{81} & & & & & \\
\hline
3+1-7+5-4+2 & +4 & +5 & -1 & +7 & -6 & +1 & +2 & -3 & +7 \\
 & +4 & +\frac{4}{3} & -\frac{28}{3} & +\frac{20}{3} & -\frac{16}{3} & +\frac{8}{3} & & & \\
\hline
 & & +\frac{11}{3} & +\frac{25}{3} & +\frac{1}{3} & -\frac{2}{3} & -\frac{5}{3} & +2 & -3 & +7 \\
 & & +\frac{11}{3} & +\frac{11}{9} & -\frac{77}{9} & +\frac{55}{9} & -\frac{44}{9} & +\frac{22}{9} & & \\
\hline
 & & & +\frac{64}{9} & +\frac{80}{9} & -\frac{61}{9} & +\frac{29}{9} & -\frac{4}{9} & -3 & +7 \\
 & & & +\frac{64}{9} & +\frac{64}{27} & -\frac{448}{27} & +\frac{320}{27} & -\frac{256}{27} & +\frac{128}{27} & \\
\hline
 & & & & +\frac{176}{27} & +\frac{265}{27} & -\frac{233}{27} & +\frac{244}{27} & -\frac{209}{27} & +7 \\
 & & & & +\frac{176}{27} & +\frac{176}{81} & -\frac{1232}{81} & +\frac{880}{81} & -\frac{704}{81} & +\frac{352}{81} \\
\hline
 & & & & & +\frac{619}{81} & +\frac{533}{81} & -\frac{148}{81} & +\frac{77}{81} & +\frac{215}{81}
\end{array}$$

(2) Synthetic polynomial division

$$\begin{array}{r|rrrr:rrrrr}
 & +4 & +5 & -1 & +7 & -6 & +1 & +2 & -3 & +7 \\
-\frac{1}{3} & & -\frac{4}{3} & -\frac{11}{9} & -\frac{64}{27} & -\frac{176}{81} & & & & \\
+\frac{7}{3} & & & +\frac{28}{3} & +\frac{77}{9} & +\frac{448}{27} & +\frac{1232}{81} & & & \\
-\frac{5}{3} & & & & -\frac{20}{3} & -\frac{55}{9} & -\frac{320}{27} & -\frac{880}{81} & & \\
+\frac{4}{3} & & & & & +\frac{16}{3} & +\frac{44}{9} & +\frac{256}{27} & +\frac{704}{81} & \\
-\frac{2}{3} & & & & & & -\frac{8}{3} & -\frac{22}{9} & -\frac{128}{27} & -\frac{352}{81} \\
\hline
 & +4 & +\frac{11}{3} & +\frac{64}{9} & +\frac{176}{27} & +\frac{619}{81} & +\frac{533}{81} & -\frac{148}{81} & +\frac{77}{81} & +\frac{215}{81}
\end{array}$$

(3) Convolution polynomial division

$$\begin{bmatrix} +4 \\ +5 \\ -1 \\ +7 \\ -6 \\ +1 \\ +2 \\ -3 \\ +7 \end{bmatrix} = \left[\begin{array}{cccc:ccccc} +3 & & & & & & & & \\ +1 & +3 & & & & & & & \\ -7 & +1 & +3 & & & & & & \\ +5 & -7 & +1 & +3 & & & & & \\ \hdashline -4 & +5 & -7 & +1 & 1 & & & & \\ +2 & -4 & +5 & -7 & & 1 & & & \\ & +2 & -4 & +5 & & & 1 & & \\ & & +2 & -4 & & & & 1 & \\ & & & +2 & & & & & 1 \end{array}\right] \begin{bmatrix} +\frac{4}{3} \\ +\frac{11}{9} \\ +\frac{64}{27} \\ +\frac{176}{81} \\ +\frac{619}{81} \\ +\frac{533}{81} \\ -\frac{148}{81} \\ +\frac{77}{81} \\ +\frac{215}{81} \end{bmatrix}$$

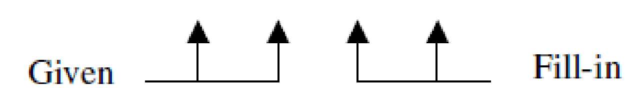
(4) Convolution polynomial division (Separated form)

$$\begin{bmatrix} +4 \\ +5 \\ -1 \\ +7 \end{bmatrix} = \begin{bmatrix} +3 & & & \\ +1 & +3 & & \\ -7 & +1 & +3 & \\ +5 & -7 & +1 & +3 \end{bmatrix} \begin{bmatrix} +\frac{4}{3} \\ +\frac{11}{9} \\ +\frac{64}{27} \\ +\frac{176}{81} \end{bmatrix}$$

$$\begin{bmatrix} -6 \\ +1 \\ +2 \\ -3 \\ +7 \end{bmatrix} = \begin{bmatrix} -4 & +5 & -7 & +1 \\ +2 & -4 & +5 & -7 \\ & +2 & -4 & +5 \\ & & +2 & -4 \\ & & & +2 \end{bmatrix} \begin{bmatrix} +\frac{4}{3} \\ +\frac{11}{9} \\ +\frac{64}{27} \\ +\frac{176}{81} \end{bmatrix} + \begin{bmatrix} +\frac{619}{81} \\ +\frac{533}{81} \\ -\frac{148}{81} \\ +\frac{77}{81} \\ +\frac{215}{81} \end{bmatrix}$$

(5) Convolution polynomial division (Template form)

$$\begin{bmatrix} +4 & +3 & +\frac{4}{3} & \circ \\ +5 & +1 & +\frac{11}{9} & \circ \\ -1 & -7 & +\frac{64}{27} & \circ \\ +7 & +5 & +\frac{176}{81} & \circ \\ -6 & -4 & \circ & +\frac{619}{81} \\ +1 & +2 & \circ & +\frac{533}{81} \\ +2 & \circ & \circ & -\frac{148}{81} \\ -3 & \circ & \circ & +\frac{77}{81} \\ +7 & \circ & \circ & +\frac{215}{81} \end{bmatrix}$$



Comparing of those approaches in this typical example, we found that the convolution polynomial division is much simple and effective. The desired quotient and remainder are readily determined without computing any intermediate data as in the familiar classical longhand polynomial division and synthetic polynomial division [3,4].

REFERENCES

1. *B. Dayton, "Polynomial GCDs by linear algebra," Theory of Equations, Northeastern Illinois University, www.neiu.edu/~bhdayton, 2004.*

2. *F. C. Chang, "Polynomial division by convolution," Applied Mathematics E-Notes, vol. 11, pp. 249-254, 2011.*

3. *L. Zhou, "Short division of polynomials," The college Mathematics Journal, **40** (2009), pp. 44-46.*

4. *D. Bini and V. Pan, "Polynomial division and its computational complexity," Journal of Complexity**, 2** (1986), pp. 179-203.*