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SUB COMPATIBLE AND SUB SEQUENTIALLY CONTINUOUS MAPS IN PROBABILISTIC METRIC SPACE

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ABSTRACT

The present paper introduces the new concepts of sub compatibility and sub sequential continuity in Probabilistic metric spaces which are weaker than occasionally weak compatibility and reciprocal continuity. We also establish a common fixed point theorem four maps using sub compatibility and sub sequential continuity.

KEYWORDS: Probabilistic Metric spaces, weak commuting mapping, compatible mappings, common fixed point

1. Introduction

In 1942, Menger [5] was first who thought about distance distribution function in metric space and introduced the concept of probabilistic metric space. He replaced distance function d(x, y), the distance between two point x,y by distance distribution function Fx,y(t) where the value of Fx,y(t) is interpreted as probability that the distance between x, y is less than t, t > 0. The study of fixed point theorem in probabilistic metrics space is useful in the study of existence of solution of operator equation in probabilistic metric space probabilistic functional analysis.

PM space has nice topological properties. Many different topological Structures may be defined on a PM space. The one that has received the most attention to date is the strong topology and it is the principle tool of this study. The convergence with respect to this topology is called strong convergence.

Schweizer and Sklar [1], developed the study of fixed point theory in probabilistic metric spaces. In 1966, Sehgal [10] initiated the study of contraction mapping theorem in probabilistic metric spaces. Several interesting and elegant result have been proved by various author in probabilistic metric spaces. In 2005, Mihet [2] proved a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. The purpose of the present paper is to prove a common fixed point theorem for six mappings via pointwise R-weakly commuting mappings in probabilistic metric spaces satisfying contractive type implicit relations. This generalizes several known results in the literature including those of Kumar and Pant [12], Kumar and Chugh [11] and others.

2. MATERIALS AND EXPOSITIONS.

Definition 2.1[12] A distribution function is a non-decreasing function $F : R \rightarrow R+$ that satisfy $\inf F(t) : t \in R = 0$ and $\sup F(t) : t \in R = 1$ and is continuous.

Δ_+ is the set of all distribution function defined on $[\infty, \infty]$.For $a \geq 0, H_a$ is an element of Δ_+ defined by

$$H_a(t) = \begin{cases} 0 & \text{if } t \leq a \\ 1 & \text{if } t > a \end{cases}$$

If X is a non-empty set, a mapping $F : X \times X \rightarrow \Delta_+$ is called a probabilistic distance on X and the value of F at $(x, y) \in X \times X$ is denoted by $F_{x,y}$

Definition 2.2[8] If F is a probabilistic distance on X, the pair (X, F) is called probabilistic Metric space (briefly PM space) if the following condition are satisfied:
(PM1) $F_{x,y}(t) = 1$ iff $x = y$
(PM2) $F_{x,y}(0) = 0$
(PM3) $F_{x,y}(t) = F_{y,x}(t)$
(PM4) If $F_{x,y}(t) = 1$ and $F_{y,z}(S) = 1$, then $F_{x,z}(t + s) = 1$ for all $x, y, z \in X$ and $s, t \geq 0$. every metric space can always realized as a PM space by considering

$F : X \times X \rightarrow \Delta_+$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$.

Definition 2.3[7] A triangle function is a binary operation τ on $\Delta_{+,}$ $(\tau : \Delta_+ \times \Delta_+ \rightarrow \Delta_+)$ which is commutative, associative non-decreasing at each place, and has H_0 as identity.

Definition 2.4[11] A triangle norm (briefly t-norm) is a binary operation Δ on [0, 1] which is commutative, associative non-decreasing with $\Delta(a, 1) = a$ for all $a \in [0, 1]$. There are four basic t-norm as follows:

(i) The minimum $t - \text{norm}, \Delta_m$ is defined by
$\Delta_m(x, y) = \min(x, y)$.

(ii) The product t-norm, Δp is defined by
$\Delta p(x, y) = x.y$.

(iii) The Lukasiecz t-norm, Δ_L is defined by
$\Delta_L(x, y) = \max(x + y - 1, 0)$

(iv) The weakest t-norm, the drastic product ΔD, is defined by

$$\Delta_D(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.5[8]A Menger space is a tripled (X, F, Δ) where (X, F) is a PM space and Δ is a t-norm such that the inequality $F_{x,z}(t + s) \geq \Delta(F_{x,y}(t), F_{y,z}(s))$ holds for all x, y, z X and $t, s \geq 0$.

Definition 2.6[7] A sequence x_n in a Menger space (X, F, Δ) is said to be converges to a point x in X iff for each $\varepsilon > 0$ and $t \in (0, 1)$, there exist is an integer $M(\varepsilon) \in N$ such that $F_{x_n,x}(\varepsilon) > 1 - t$ for all $n \geq M(\varepsilon)$.
Definition 2.7[11] The sequence $\{\{x_n\}\}$ is said to be Cauchy sequence if for each $\varepsilon > 0$ and t ε (0, 1), there exist is an integer $M(\varepsilon) \in N$ such that $F_{x_n,x_m}(\varepsilon) > 1 - t$ for all $n, m \geq M(\varepsilon)$.

Definition 2.8[8] A Menger space (X, F, Δ) is said to be complete, if every Cauchy sequence in X converges to a point x in X.

Definition 2.9[12] Two self mapping f and g of a Menger space (X, F, Δ) are said to be weakly commuting if $F_{fgx,gfx}(t) \geq F_{fx,gx}(t)$ for each x in X and $t > 0$.

Definition 2.10[11] Two self mapping f and g of a Menger space (X, F, Δ) are said to be point wise R-weakly commuting if $F_{fgx,gfx}(t) \geq F_{fx,gx}(t/R)$ for each x in X and $t > 0$.

Definition 2.11[12] Two self mapping f and g of a Menger space (X, F, Δ) are said to be compatible iff $F_{fgx_n, gfx_n}(t) \to 1$ for all $t > 0$,whenever $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to z \in X$ for some $z \in X$.

Definition 2.12[1] Two self maps A and B on a set X are said to be occasionally weakly compatible if and only if there is a point which is a coincidence point of A and B at which A and B commute. i.e., there exists a point $x \in X$ such that $Ax = Bx$ and $ABx = BAx$.

Definition 2.13[8] Two self maps A and B on an Menger-space (X, F) are said sub compatible if and only if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z, z \in X$ and which satisfy $\lim_{n\to\infty} F_{(ABx_n, BAx_n)}(t) = 1$ and for all $t > 0$

Definition 2.14[11] Two self maps A and S on a Menger-space are called reciprocal continuous if $\lim_{n\to\infty} AS\, x_n = At$ and $\lim_{n\to\infty} SA\, x_n = St$ for some ϵ X . whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} A\, x_n = At$ and $\lim_{n\to\infty} S\, x_n = St$ for $t \epsilon X$

Definition 2.15[7] Two self maps A and S on a Probabilistic metric space are said to be sub sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} A\, x_n = \lim_{n\to\infty} S\, x_n = t \epsilon X$ and satisfy $\lim_{n\to\infty} AS\, x_n = At$ and $\lim_{n\to\infty} S\, Ax_n = St$.

Lemma 2.1 Let (X,F) be an Menger metric space so if there exist k ∈ (0,1) such that

$P_{x,y}(kt) \geq P_{x,y}(t)$ for all $x, y \in X$,. Then, x = y

Implicit relation: Let Φ denote the set of all continuous functions $[0,1]^4 \to R$ satisfying Φ_1: Φ is non-increasing in second and third argument and Φ_2: $\Phi\ (u, v, v, v,) \geq 0\ \ u \geq v$ for $u, \in [0,1]$

3. <u>THEOREM AND DISSCUSSION</u>

Theorem 3.1: Let f,g,h and k be four self maps on a complete probabilistic metric space (X,F).If the pair (f,h) and (g,k) are subcompatible and sub sequentially continuous then

(i) f and h have a coincidence point

(ii) g and k have a coincidence point.

Further if –

$$\varphi\left(F_{(fx,gy)}\ (kt), \frac{F_{(hx,ky)}(t) + F_{(fx,hx)}(t)}{2}, \frac{F_{(gy,ky)}(t) + F_{(hx,gy)}(t)}{2}, F_{(ky,fx)}\ (t)\right) \geq 0$$

For all k ∈ (0,1) x,y ∈ X, t > 0 and $\varphi \in \Phi$ Then f,g,h and k have a unique common fixed point in k.

Proof: Since the pairs (f,h) and (g,k) are sub compatible and sub sequentially continuous.

There exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that –

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = u, \text{for some } u \in X$

And which satisfy –

$$\lim_{n\to\infty} F_{(fhx_{n,hfx_n})}(t) = F_{(fu,hu)}(t) = 1$$

$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} ky_n = v, \text{for some } v \in X$

And which satisfy –

$$\lim_{n\to\infty} F_{(gky_{n,kgy_n})}(t) = F_{(gv,kv)}(t) = 1$$

Therefore $fu = hu$ and $gv = kv$

That is; u is the coincidence point of f and h and v is the coincidence point of g and k

Now,by (3) for $x = x_n$ and $y = y_n$, we get

$$\varphi\begin{pmatrix} F_{(fx_n,gy_n)}(kt), \frac{F_{(hx_n,ky_n)}(t)+F_{(fx_n,hx_n)}(t)}{2}, \\ \frac{F_{(gy_n,ky_n)}(t)+F_{(hx_n,gy_n)}(t)}{2}, F_{(ky_n,fx_n)}(t) \end{pmatrix} \geq 0$$

Letting $n \to \infty$,

$$\varphi\left(F_{(u,v)}(kt), \frac{F_{(u,v)}(t)+1}{2}, \frac{1+F_{(u,v)}(t)}{2}, F_{(u,v)}(t)\right) \geq 0$$

Since φ is non-decreasing in second and third argument ,therefore

$$\varphi\left(F_{(u,v)}(kt), F_{(u,v)}(t), F_{(u,v)}(t), F_{(u,v)}(t)\right) \geq 0$$

by using the property of φ

$$F_{(u,v)}(kt) \geq F_{(u,v)}(t)$$

Therefore by lemma 2.1

$$u = v$$

Again by (3) for $x = u$ and $y = y_n$

$$\varphi\begin{pmatrix} F_{(fu,gy_n)}(kt), \frac{F_{(hu,ky_n)}(t)+F_{(fu,hu)}(t)}{2}, \\ \frac{F_{(gy_n,ky_n)}(t)+F_{(hu,gy_n)}(t)}{2}, F_{(ky_n,fu)}(t) \end{pmatrix} \geq 0$$

Letting $n \to \infty$,

$$\varphi\left(F_{(fu,v)}(kt), \frac{F_{(fu,v)}(t)+1}{2}, \frac{1+F_{(fu,v)}(t)}{2}, F_{(fu,v)}(t)\right) \geq 0$$

Since φ is non-decreasing in second and third argument ,therefore

$$\varphi\left(F_{(fu,v)}(kt), F_{(fu,v)}(t), F_{(fu,v)}(t), F_{(fu,v)}(t)\right) \geq 0$$

by using the property of φ

$$F_{(fu,v)}(kt) \geq F_{(fu,v)}(t)$$

Therefore by lemma 2.1

$$fu = v = u$$

Therefore $u = v$ is a common fixed point of f, g, h, and k

For uniqueness, let $w \neq u$ be another fixed point of f,g,h and k.

Therefore from (3),we have

$$\varphi\begin{pmatrix} F_{(fu,gw)}(kt), \dfrac{F_{(hu,kw)}(t) + F_{(fu,hw)}(t)}{2}, \\ \dfrac{F_{(gw,kw)}(t) + F_{(hu,gw)}(t)}{2}, F_{(kw,fu)}(t) \end{pmatrix} \geq 0$$

That is; $\varphi\left(F_{(fu,gw)}(kt), \frac{F_{(fu,gw)}(t)+1}{2}, \frac{1+F_{(fu,gw)}(t)}{2}, F_{(gw,fu)}(t)\right) \geq 0$

Since φ is non-decreasing in second and third argument ,therefore

$$\varphi\left(F_{(fu,gw)}(kt), F_{(gw,fu)}(t), F_{(gw,fu)}(t), F_{(gw,fu)}(t)\right) \geq 0$$

by using the property of φ

$$F_{(fu,gw)}(kt) \geq F_{(gw,fu)}(t)$$

Therefore by lemma 2.1 ,

$$fu = gw$$

and hence the theorem.

Corollary 3.2: Let f and h be self maps on an probabilistic metric space (X,F). such that the pair (f,h) is sub compatible and sub sequentially continuous, then f and h have a coincidence point,

Further, if

$$\varphi\left(F_{(fx,fy)}(kt), \frac{F_{(hx,hy)}(t) + F_{(fx,hx)}(t)}{2}, \frac{F_{(fy,hy)}(t) + F_{(hx,fy)}(t)}{2}, F_{(hy,fx)}(t)\right) \geq 0$$

For all $k \in (0, 1)$ $x,y \in X$, $t > 0$ and $\varphi \in \Phi_4$

Then f and h have a unique common fixed point in X.

Corollary 3.3: Let f,g and h be self maps on an probabilistic metric space (X,F), such that the pair (f,h) and (f,h) are sub compatible and sub sequentially continuous, then f and h have a coincidence point,

g and h have a coincidence point

Further, if

$$\varphi\left(F_{(fx,fy)}(kt), \frac{F_{(hx,hy)}(t)+F_{(fx,hx)}(t)}{2}, \frac{F_{(gy,hy)}(t)+F_{(hx,gy)}(t)}{2}, F_{(hy,fx)}(t)\right) \geq 0$$

For all k ∈ (0,1) x,y ∈ X, t > 0 and $\varphi \in \Phi_4$

Then f,g and h have a unique commom fixed point in X.

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