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LIMIT INFIMUM RESULTS FOR SUBSEQUENCES OF DELAYED RANDOM SUMS AND RELATED BOUNDARY CROSSING PROBLEM

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ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. strictly positive stable random variables with exponent α, $0 < \alpha < 1$. We study a non-trivial limit behavior of linearly normalized subsequences of delayed random sums and extended to boundary crossing problem.

Keywords: Law of iterated logarithm, Delayed random sums, Domain of attraction, Stable law.

1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.s.) and let $S_n = \sum_{k=1}^{n} X_k$, $\forall\, n \geq 1$. Set $T_{a_n} = \sum_{i=n+1}^{n+a_n} X_i = S_{n+a_n} - S_n$, where a_n, $\forall\, n \geq 1$ be a non-decreasing function of a positive integers of n such that, $0 < a_n \leq n$, for all n and $\frac{a_n}{n} \sim b_n$, where b_n is non-increasing. Let $\gamma_n = \left\{\log \frac{n}{a_n} + \log\log n\right\}$. The sequence $\{T_{a_n}, n \geq 1\}$ is called a (forward) delayed sum sequence (See Lai (1973)).

Now parallel to the delayed sums T_{a_n}, we introduce delayed random sums as, $M_{N_n} = \sum_{j=n+1}^{n+N_n} X_j = S_{n+N_n} - S_n$, where $\{N_n, n \geq 1\}$ be a sequence of positive r.v.s. independent of $\{X_n, n \geq 1\}$ such that $\lim_{n\to\infty} \frac{N_n}{n} = 1$ almost surely.

When X_n's are i.i.d. symmetric stable r.v.s., Chover (1966) established the law of iterated logarithm (LIL) for (S_n), by normalizing in the power. Divanji and Vasudeva (1989) extended the same to the domain of partial attraction of a semi stable law with exponent α, $0 < \alpha < 2$. When Gut (1986) established the classical LIL for geometrically fast increasing subsequences of (S_n). Torrang (1987) extended the same to random subsequences.

When variance is finite, Lai (1973) had studied the behavior of classical LIL for properly normalized sums T_{a_n}, at different values of a_n's. He proved that these results are entirely different from LIL for partial sums . For independent but not identically distributed strictly positive stable r.v.s. Vasudeva and Divanji (1993) studied a non-trivial limit behavior of delayed sums T_{a_n} .

When r.v.s. are i.i.d. non-negative, which are in the domain of attraction of a completely asymmetric positive stable law, with exponent α, $0 < \alpha < 1$, Gooty Divanji and Raviprakash (2016) studied Chover's form of LIL for normalized delayed random sums $\{M_{N_n}, n \geq 1\}$.

Observations made by Gut (1986) motivated us to examine whether limit infimum for properly normalized delayed random sums $\left\{M_{N_{n_k}}, k \geq 1\right\}$ can be studied, for the i.i.d. strictly positive stable r.v.s. We answer in affirmative and extend the same to number of boundary crossings associated with this LIL.

Throughout the paper C, ε, η, δ, τ, ρ and k with or without a suffix or super suffix stand for positive constants with k and n confined to be positive integers. i.o. and a.s. stand for infinitely often and almost surely respectively.

In the next section, we present some known results. In section 3, we established a non-trivial limit behavior of linearly normalized subsequences of delayed random sums $\left\{M_{N_{n_k}}, k \ge 1\right\}$ and in the last section, we extend the study to boundary crossing problem.

2. SOME KNOWN RESULTS

Lemma 2.1 (Extended Borel-Cantelli Lemma)

Let (E_n) be a sequence of events with a common probability space $(\Omega, \Im, P)$.

If (i) $\sum_{n=1}^{\infty} P(E_n) = \infty$ and

(ii) $$\operatorname*{Lim\,inf}_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{s=1}^{n} P(E_k \cap E_s)}{\left(\sum_{k=1}^{n} P(E_k)\right)^2} \le C$$ then

$P(E_n \text{ i.o.}) \ge C^{-1}$.

For proof, see Spitzer (1964, Lemma p3, p.317)

Lemma 2.2

Let X_1 be a positive stable r.v. with characteristic function

$$E(\exp\{iuX_1\}) = \exp\left\{-|u|^{\alpha}\left(1 - \frac{iu}{|u|}\tan\left(\frac{\pi\alpha}{2}\right)\right)\right\},$$

$0 < \alpha < 1$. Then, as $x \to 0$,

$$P(X_1 \le x) \sim \frac{x^{\frac{\alpha}{2(1-\alpha)}}}{\sqrt{2\pi\alpha\, B(\alpha)}} \exp\left\{- B(\alpha)\, x^{-\frac{\alpha}{(1-\alpha)}}\right\},$$ where

$$B(\alpha) = (1-\alpha)\, \alpha^{\frac{\alpha-1}{\alpha}} \left(\operatorname{Cos}\frac{\pi\alpha}{2}\right)^{\frac{1}{\alpha-1}}$$

For proof see Vasudeva and Divanji (1989).

Lemma 2.3

Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. non-negative r.v.s, which are in the domain of attraction of a completely asymmetric positive stable law, with index α, $0 < \alpha < 1$ and let $\{N_n, n \ge 1\}$ be a sequence of positive r.v.s. independent of $\{X_n, n \ge 1\}$ such that $\operatorname*{Lim}_{n\to\infty} \frac{N_n}{n} = 1$ a.s. Then,

$$\operatorname*{Lim\,inf}_{n\to\infty} \left(\frac{M_{N_n}}{\beta(N_n)}\right) = 1 \text{ a.s.}$$

The proof follows similar lines of Theorem 2 of Gooty Divanji and K N Raviprakash(2016) and hence details are omitted.

3 A NON-TRIVIAL LIMIT BEHAVIOR FOR LINEARLY DELAYED RANDOM SUMS

Theorem 3.1

Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. strictly positive stable r.v.s. with indext α, $0 < \alpha < 1$. Let $\{n_k\}$ be a integer subsequence such that $\operatorname*{Lim\,sup}_{k\to\infty} \frac{n_k}{n_{k+1}} < 1$ and let $\left\{N_{n_k}, k \ge 1\right\}$ be a subsequence of r.v.s independent of $\{X_n, n \ge 1\}$ such that $\frac{N_{n_k}}{n_k} \to 1$ a.s..

Then $\liminf_{k\to\infty} \frac{M_{N_{n_k}}}{\beta(N_{n_k})} = \varepsilon^*$ a.s.,

where $\varepsilon^* = \sup\left\{\varepsilon_1 > 0 : \sum_{k \ge k_0} (\log n_k)^{-\varepsilon_1^{\frac{\alpha}{\alpha-1}}} < \infty\right\}$,

for some $k_0 > 0$,

$$\beta(N_{n_k}) = \theta_\alpha N_{n_k}^{\frac{1}{\alpha}} \left(\log \frac{n_k}{N_{n_k}} + \log\log n_k\right)^{\frac{\alpha-1}{\alpha}},$$

$$\theta_\alpha = (B(\alpha))^{\frac{1-\alpha}{\alpha}},$$

$B(\alpha) = (1-\alpha)\alpha^{\frac{\alpha-1}{\alpha}} \left(\operatorname{Cos}\frac{\pi\alpha}{2}\right)^{\frac{1}{\alpha-1}}$ and

$$M_{N_{n_k}} = \sum_{i=1}^{n+N_{n_k}} X_i = S_{n_k+N_{n_k}} - S_{n_k}.$$

Proof

Equivalently, we show that, for any $\varepsilon > 0$,

$$P\left(M_{N_{n_k}} \le (\varepsilon^* - \varepsilon)\beta(N_{n_k}) \text{ i.o.}\right) = 0 \quad (1)$$

and

$$P\left(M_{N_{n_k}} \le (\varepsilon^* + \varepsilon)\beta(N_{n_k}) \text{ i.o.}\right) = 1 \quad (2)$$

From the condition, $\frac{N_{n_k}}{n_k} \to 1$ a.s. as $k \to \infty$ implies that, for any $\delta \in (0,1)$, we have,

$u_{n_k} \le N_{n_k} \le v_{n_k}$ a.s., (3)

where, $u_{n_k} = (1-\delta)n_k$ and $v_{n_k} = (1+\delta)n_k$,

Consequently, one can notice that, as X_n's are i.i.d. strictly positive stable r.v.s., we have,

$M_{u_{n_k}} \le M_{N_{n_k}} \le M_{v_{n_k}}$ a.s. (4)

By condition, $\frac{N_{n_k}}{n_k} \to 1$ a.s. as $k \to \infty$ and (3), we can observe that,

$\frac{\beta(N_{n_k})}{\beta(n_k)} \to 1$ a.s.

as $k \to \infty$, which implies that there exits some $\delta_1 > 0$, such that,

$(1-\delta_1) \le \frac{\beta(N_{n_k})}{\beta(n_k)} \le (1+\delta_1)$. (5)

Using (4) and (5), it is enough to prove (1) and (2) as,

$$P\left(M_{u_{n_k}} \le (\varepsilon^* - \varepsilon)(1+\delta_1)\beta(n_k) \text{ i.o.}\right) = 0 \quad (6)$$

where, $M_{u_{n_k}} = S_{n_k+u_{n_k}} - S_{n_k}$ and

$$P\left(M_{v_{n_k}} \le (\varepsilon^* + \varepsilon)(1-\delta_1)\beta(n_k) \text{ i.o.}\right) = 1 \quad (7)$$

where, $M_{v_{n_k}} = S_{n_k+v_{n_k}} - S_{n_k}$

The condition $\limsup_{k\to\infty} \frac{n_k}{n_{k+1}} < 1$

implies that, there exits some constant b>1 such that, $n_{k+1} \ge b n_k$, for some k large (8)

From the fact that, X_n's are strictly positive stable r.v.s. with exponent α, $0 < \alpha < 1$, we have $\frac{M_{u_{n_k}}}{u_{n_k}^{\frac{1}{\alpha}}}$ and X_1 are identically distributed and hence,

$$P\left(M_{u_{n_k}} \le (\varepsilon^* - \varepsilon)(1+\delta_1)\beta(n_k)\right) = P\left(X_1 \le \frac{(\varepsilon^*-\varepsilon)(1+\delta_1)\beta(n_k)}{u_{n_k}^{\frac{1}{\alpha}}}\right).$$

In view of definition of $\beta(N_{n_k})$, we can notice that, there exits some constant $C_1 > 0$ such that,

$\frac{(\varepsilon^*-\varepsilon)(1+\delta_1)\beta(n_k)}{u_{n_k}^{\frac{1}{\alpha}}} = C_1\theta_\alpha(\varepsilon^*-\varepsilon)(\log\log n_k)^{\frac{\alpha-1}{\alpha}}$. Taking x as $C_1\theta_\alpha(\varepsilon^*-\varepsilon)(\log\log n_k)^{\frac{\alpha-1}{\alpha}}$ in Lemma 2.2, we can find some k_1 (>0) and C_2 (> 0) such that,

$$P\left(X_1 \le \frac{(\varepsilon^*-\varepsilon)(1+\delta_1)\beta(n_k)}{u_{n_k}^{\frac{1}{\alpha}}}\right) \sim \frac{C_2}{(\log\log n_k)^{\frac{1}{2}}}\exp\left\{-(\varepsilon^*-\varepsilon)^{\frac{\alpha}{\alpha-1}}\log\log n_k\right\}$$

$$\sim \frac{C_2}{(\log\log n_k)^{\frac{1}{2}}(\log n_k)^{(\varepsilon^*-\varepsilon)^{\left(\frac{\alpha}{\alpha-1}\right)}}}$$

$$\le \frac{C_2}{(\log n_k)^{(\varepsilon^*-\varepsilon)^{\frac{\alpha}{\alpha-1}}}}.$$

From the fact that $0 < \varepsilon^* - \varepsilon < \varepsilon^*$ for some ε sufficiently small and by the definition of ε^* and also by (8), we have, for some k_1 (> 0),

$$\sum_{k\ge k_1} P\left(M_{u_{n_k}} \le (\varepsilon^*-\varepsilon)\ (1+\delta_1)\ \beta(n_k)\right) \le C_2\sum_{k\ge k_1}\frac{1}{(\log n_k)^{(\varepsilon^*-\varepsilon)^{\frac{\alpha}{\alpha-1}}}} < \infty.$$

So that (6) follows by Borel - Cantelli Lemma. Consequently (1) follows from (6).

To prove (7), we prove that, for some d > 0,

$$P\left(M_{v_{n_k}} \le (\varepsilon^*+\varepsilon)(1-\delta_1)\beta(n_k) \text{ i.o.}\right) \ge d > 0. \quad (9)$$

Define the events

$$D_k=\left\{0<M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_k\right)\right\}.$$

Then we have,

$$P\left(D_k\right)=P\left(0<M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\ \left(1-\delta_1\right)\beta\left(n_k\right)\right).$$

From the fact that, X_n's are strictly positive stable r.v.s. with exponent $\alpha, 0<\alpha<1$, we have, $\frac{M_{v_{n_k}}}{v_{n_k}^{\frac{1}{\alpha}}}$ and X_1 are identically distributed and hence,

$$P\left(D_k\right)=P\left(X_1\le\frac{\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_k\right)}{v_{n_k}^{\frac{1}{\alpha}}}\right)\quad(10)$$

Again by the definition of $\beta\left(N_{n_k}\right)$, we observe that, there exits some constant $C_3>0$ such that,

$$\frac{\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_k\right)}{v_{n_k}^{\frac{1}{\alpha}}}=C_3\theta_\alpha\left(\varepsilon^*+\varepsilon\right)\left(\log\log n_k\right)^{\frac{\alpha-1}{\alpha}}$$

. Again taking x as $C_3\theta_\alpha\left(\varepsilon^*+\varepsilon\right)\ \left(\log\log\ n_k\right)^{\frac{\alpha-1}{\alpha}}$ in Lemma 2.2, we can find some k_2 (> 0) and C_4 (> 0) such that,

$$P\left(D_k\right)=P\left(X_1\le\frac{\left(\varepsilon^*+\varepsilon\right)\ \left(1-\delta_1\right)\ \beta\left(n_k\right)}{v_{n_k}^{\frac{1}{\alpha}}}\right)\sim\frac{C_4}{\left(\log\log n_k\right)^{\frac{1}{2}}}\exp\left\{-\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}\log\log n_k\right\}$$

$$\sim\frac{C_4}{\left(\log\log n_k\right)^{\frac{1}{2}}\left(\log n_k\right)^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}}$$

$$\ge\frac{C_4}{\left(\log n_k\right)^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}}\quad(11)$$

From the fact that $0<\varepsilon^*+\varepsilon<\varepsilon^*$ for some ε sufficiently small and by the definition of ε^* we have,

$$\sum_{k\ge k_2}P\left(0<M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_k\right)\right)\ge C_4\sum_{k\ge k_2}\frac{1}{\left(\log n_k\right)^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}}=\infty$$

and hence $P\left(D_k\right)=\infty$.

Now let $s>t_k$, where $t_k=k+\tau(\log k)$, for some $\tau>1$. Then,

$$P\left(D_k\cap D_s\right)=P\begin{pmatrix}0<M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\ \left(1-\delta_1\right)\ \beta\left(n_k\right)\cap\\ 0<M_{v_{n_s}}\le\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_s\right)\end{pmatrix}.$$

Observe that,

$$\left\{\begin{matrix}0<M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\ \left(1-\delta_1\right)\ \beta\left(n_k\right),\\ 0<M_{v_{n_s}}\le\left(\varepsilon^*+\varepsilon\right)\ \left(1-\delta_1\right)\ \beta\left(n_s\right)\end{matrix}\right\}$$

$$\subseteq\left\{\begin{matrix}0<M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_k\right),\\ 0<M_{v_{n_s}}-M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_s\right)\end{matrix}\right\}$$

and hence,

$$P\left(D_k\cap D_s\right)\le P\begin{pmatrix}0<M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_k\right),\\ 0<M_{v_{n_s}}-M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_s\right)\end{pmatrix}.$$

As $M_{v_{n_k}}$ and $M_{v_{n_s}}-M_{v_{n_k}}$ are independent, we get,

$$P\left(D_k\cap D_s\right)=P\left(D_k\right)P\left(M_{v_{n_s}}-M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_s\right)\right)\quad(12)$$

Again using the fact that, X_n's are strictly positive stable r.v.s. with exponent $\alpha, 0<\alpha<1$, we have, $\frac{M_{v_{n_s}}-M_{v_{n_k}}}{v_{n_s}^{\frac{1}{\alpha}}-v_{n_k}^{\frac{1}{\alpha}}}$ and X_1 are identically distributed and hence,

$$P\left(M_{v_{n_s}}-M_{v_{n_k}}\le\left(\varepsilon^*+\varepsilon\right)\ \left(1-\delta_1\right)\ \beta\left(n_s\right)\right)=P\left(X_1\le\frac{\left(\varepsilon^*+\varepsilon\right)\left(1-\delta_1\right)\beta\left(n_k\right)}{\left(v_{n_s}-v_{n_k}\right)^{\frac{1}{\alpha}}}\right).$$

Following the steps similar to those used to get (11), we can find some constant $C_5(>0)$ and a $k_3(>0)$ and such that, for all $k\ge k_3$,

$$P\left(M_{v_{n_s}} - M_{v_{n_k}} \le \left(\varepsilon^* + \varepsilon\right)\left(1-\delta_1\right)\beta\left(n_s\right)\right) \le \frac{C_5}{\left(\log n_s\right)^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}}$$

. Notice that, there exists some constant C_6 (>C_5) such that

$$P\left(M_{v_{n_s}} - M_{v_{n_k}} \le \left(\varepsilon^* + \varepsilon\right)\left(1-\delta_1\right)\beta\left(n_s\right)\right) \le C_6 P\left(D_s\right)$$

. From (12), we have,

$$P\left(D_k \cap D_s\right) \le C_6 P\left(D_k\right) P\left(D_s\right), \qquad (13)$$

for some $s > t_k$, where, $t_k = k + \tau(\log k)$ for some τ >1.

Now for $(k+1) \le s \le t_k$, where $t_k = k + \tau(\log k)$, for some τ >1, observe that, $M_{v_{n_s}}$ and $M_{v_{n_k}}$ are independent and using the inequality $s \ge k+1$ we have,

$P\left(D_k \cap D_s\right) \le P\left(D_k\right)$ which implies,

$$\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P\left(D_k \cap D_s\right) \le \sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P\left(D_k\right) \le \sum_{k=1}^{n} \tau(\log k) P\left(D_k\right) \qquad (14)$$

Using Lemma 2.2 in (10) and hence from (14), we can find some constant C_7 (>0) such that,

$$\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P\left(D_k \cap D_s\right) \le C_7 \sum_{k=1}^{n} \frac{(\log k)}{\left(\log n_k\right)^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}} \qquad (15)$$

From (8) we have, $n_k \ge b^k n_0$, where b>1 and n_0 is fixed and hence,

$$\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P\left(D_k \cap D_s\right) \le C_7 \sum_{k=1}^{n} \frac{\log k}{k^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}} \qquad (16)$$

Using (8) in (11), we can find some constant, C_8 (>0) and a k_4 (>0) such that, for all $k \ge k_4$,

$$\sum_{k=k_4}^{n} P\left(D_k\right) \ge C_8 \sum_{k=k_4}^{n} \frac{1}{k^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}}.$$

From the definition of ε^* we have, $\varepsilon^* \ge 1$, which implies $\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}} < 1$ and hence,

$$k^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}} < k \text{ or } \sum_{k=k_4}^{n} \frac{1}{k^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}} > \sum_{k=k_4 1}^{n} \frac{1}{k}.$$

Therefore,

$$\sum_{k=k_4}^{n} P\left(D_k\right) \ge C_8 \sum_{k=k_4}^{n} \frac{1}{k^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}} > C_8 \sum_{k=k_4}^{n} \frac{1}{k} \sim \log n \qquad (17)$$

From (16) and (17), we have, there exists C_9 (>0) such that,

$$\frac{\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P\left(D_k\right)\right)^2} = \frac{\sum_{k=1}^{n} \sum_{s=k+1}^{t_k} P\left(D_k \cap D_s\right)}{\left(\sum_{k=1}^{n} P\left(D_k\right)\right)\left(\sum_{k=1}^{n} P\left(D_k\right)\right)} \le \frac{C_7 \sum_{k=1}^{n} \frac{\log k}{k^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}}}{\left(C_5 \sum_{k=1}^{n} \frac{1}{k^{\left(\varepsilon^*+\varepsilon\right)^{\frac{\alpha}{\alpha-1}}}}\right) \log n} \le \frac{C_7}{C_5} < C_9 (>0), \qquad (18)$$

it holds for $(k+1) \le s \le t_k$, $t_k = k + \tau(\log k)$, for some τ >1.

From (13), we have, $s > t_k$, $t_k = k + \tau(\log k)$, for some τ >1,

$$\sum_{k=1}^{n} \sum_{s=t_k}^{n} P\left(D_k \cap D_s\right) \le C_6 \sum_{k=1}^{n} \sum_{s=t_k}^{n} P\left(D_k\right) P\left(D_s\right) \le C_6 \left(\sum_{k=1}^{n} P\left(D_k\right)\right)\left(\sum_{s=1}^{n} P\left(D_s\right)\right)$$

$$\le C_6 \left(\sum_{k=1}^{n} P\left(D_k\right)\right)\left(\sum_{k=1}^{n} P\left(D_k\right)\right) = C_6 \left(\sum_{k=1}^{n} P\left(D_k\right)\right)^2 \qquad (19)$$

Observe that,

$$\sum_{k=1}^{n}\sum_{s=t_k}^{n} P(D_k \cap D_s) \sim$$

$$2\sum_{k=1}^{n}\sum_{s=k+1}^{t_k} P(D_k \cap D_s) \sim$$

$$2\sum_{k=1}^{n-1}\left(\sum_{s=k+1}^{t_k} P(D_k \cap D_s) \sum_{s=t_k+1}^{n} P(D_k \cap D_s)\right) \sim$$

$$2\sum_{k=1}^{n-1}\sum_{s=k+1}^{t_k} P(D_k \cap D_s) + 2\sum_{k=1}^{n-1}\sum_{s=t_k+1}^{n} P(D_k \cap D_s) . \quad (20)$$

This implies,

$$\frac{\sum_{k=1}^{n}\sum_{s=1}^{t_k} P(D_k \cap D_s)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} \sim \frac{2\sum_{k=1}^{n-1}\sum_{s=k+1}^{t_k} P(D_k \cap D_s)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} + \frac{2\sum_{k=1}^{n-1}\sum_{s=t_k+1}^{n} P(D_k \cap D_s)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2}$$

From (18), (19) and (20), one can find some constant C_{10} (> 0) such that,

$$\frac{\sum_{k=1}^{n}\sum_{s=1}^{t_k} P(D_k \cap D_s)}{\left(\sum_{k=1}^{n} P(D_k)\right)^2} \geq C_{10}(>0).$$

In view of the series $\sum_{k\geq k_2} P(D_k) = \infty$ and (19), appealing to E B C Lemma 2.1 and Hewitt - Sevage zero-one law, proof of (7) follows from $P(D_k \text{ i.o.}) = 1$ and consequently proof of (2) follows from (7). Hence proof of the theorem is completed.

Theorem 3.2

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. strictly positive stable r.v.s. with index $\alpha, 0 < \alpha < 1$. Let $\{n_k, k \geq 1\}$ be a integer subsequence such that $\operatorname{Lim\,inf}_{k\to\infty} \frac{n_k}{n_{k+1}} > 0$ and let $\{N_{n_k}, k \geq 1\}$ be a sequence of r.v.s . independent of $\{X_n, n \geq 1\}$ such that $\frac{N_{n_k}}{n_k} \to 1$ a.s. Then $\operatorname{Lim\,inf}_{k\to\infty} \frac{M_{N_{n_k}}}{\beta(N_{n_k})} = 1$ *a.s.*,

Proof: To prove the theorem, it is enough to show that, for any $\varepsilon \in (0,1)$,

$$P\left(M_{N_{n_k}} \leq (1-\varepsilon)\beta(N_{n_k}) \text{ i.o.}\right) = 0 \quad (21)$$

and

$$P\left(M_{N_{n_k}} \leq (1+\varepsilon)\beta(N_{n_k}) \text{ i.o.}\right) = 1 \quad (22)$$

We follow similar steps of Theorem 3.1, it is sufficient to prove (21) and (22) as,

$$P\left(M_{u_{n_k}} \leq (1-\varepsilon)(1+\delta_1)\beta(n_k) \text{ i.o.}\right) = 0 \quad (23)$$

where $M_{u_{n_k}} = S_{n_k+u_{n_k}} - S_{n_k}$ and

$$P\left(M_{v_{n_k}} \leq (1+\varepsilon)(1-\delta_1)\beta(n_k) \text{ i.o.}\right) = 1 \quad (24)$$

where $M_{v_{n_k}} = S_{n_k+v_{n_k}} - S_{n_k}$

The condition $\operatorname{Lim\,inf}_{k\to\infty} \frac{n_k}{n_{k+1}} > 0$ implies that, there exits some $\rho > 0$ such that,

$$n_{k+1} < \frac{n_k}{\rho}, \text{ for k large} \quad (25)$$

This in turn implies that, $\{M_{v_{n_k}}, k \geq 1\}$ is a sequence of mutually independent r.v.s.

To prove (23), it is sufficient to show (6) for $\varepsilon^* = 1$. For any $\varepsilon \in (0,1)$, by Lemma 2.3, we claim that, $\operatorname{Lim\,inf}_{k\to\infty} \frac{M_{v_{n_k}}}{\beta(n_k)} \geq \operatorname{Lim\,inf}_{k\to\infty} \frac{M_{N_{n_k}}}{\beta(N_{n_k})} = 1$ a.s. which establishes (23), for $\varepsilon^* = 1$ and consequently proof of (21) follows from (23), for $\varepsilon^* = 1$.

Now to establish (24), for $\varepsilon^* = 1$, we proceed as in Allan Gut (1986). Define,

$m_j = \min\{k : n_k > M^j\}$, where j=1,2…. and M is chosen such that $\frac{1}{\rho^2 M} < 1$. Hence using (25), we get that, $M^j \le n_{m_j} \le \frac{M^j}{\rho^2}$ and

$\rho^2 \le \frac{n_{m_{j-1}}}{n_{m_j}} \le \frac{1}{\rho^2 M}$, j=1,2….; Consequently, $\left(n_{m_j}\right)$ satisfies the condition $\operatorname{Lim\,sup}_{j\to\infty} \frac{n_{m_{j-1}}}{n_{m_j}} < 1$ of Theorem 2.1 with $\sum_{k\ge k_0}\left(\log\ n_{m_j}\right)^{-\varepsilon_1^{\frac{\alpha}{\alpha-1}}}$, for all $\varepsilon < 1$ (i.e. $\varepsilon^* \ge 1$).

Hence (24) follows from Theorem 3.1, for $\varepsilon^* = 1$. Consequently (22) follows from (24) and hence proof of Theorem is completed.

4. NUMBER OF BOUNDARY CROSSING PROBLEM

In this section, we study some boundary crossings r.v.s. related to Theorems 3.1 and 3.2.

Define for any $\varepsilon > 0$,

$$Y_k(\varepsilon) = \begin{cases} 1, & \text{if } M_{N_{n_k}} \le (\theta \pm \varepsilon)\beta\left(N_{n_k}\right) \\ 0, & \text{otherwise} \end{cases}$$

where,

$$\theta = \begin{cases} \varepsilon^*, & \text{if } (n_k) \text{ is atleast geometrically fast} \\ 1, & \text{if } (n_k) \text{ is atmost geometrically fast} \end{cases}$$

$$M_{N_{n_k}} = \sum_{i=1}^{n+N_{n_k}} X_i = S_{n_k+N_{n_k}} - S_{n_k},$$

$$\varepsilon^* = \sup\left\{\varepsilon_1 > 0 : \sum_{k\ge k_0}(\log n_k)^{-\varepsilon_1^{\frac{\alpha}{\alpha-1}}} < \infty\right\},$$

for some $k_0 > 0$. For any $\varepsilon > 0$, we have from (1), $P\left(M_{N_{n_k}} \le \left(\varepsilon^* - \varepsilon\right)\beta\left(N_{n_k}\right) \text{ i.o.}\right) = 0$

(26)

Define $N_\infty(\varepsilon) = \sum_{k\ge 1} Y_k(\varepsilon)$ and observe that $N_\infty(\varepsilon)$ is a proper r.v. in view of (26).

Let $\left\{N_{m_k}(\varepsilon)\right\}$ be the corresponding sequence of partial sums. i.e. $N_{m_k}(\varepsilon) = \sum_{k=1}^{m_k} Y_k(\varepsilon)$, where $\{m_k, k \ge 1\}$ is a subsequence of sequence. If $N_\infty(\varepsilon) = \sum_{k=1}^{\infty} Y_k(\varepsilon)$, we know that $P(N_\infty(\varepsilon) < \infty) = 1$ or $N_\infty(\varepsilon)$ is a proper r.v. of number of boundary crossings and hence it is interesting to study the existence of moments for this boundary crossing r.v.s. and obtain moments of this proper r.v. $N_\infty(\varepsilon)$ as Corollary to Theorems 3.1 and 3.2. This boundary crossing r.v.s. was studied by various authors like Slivka(1969) and Slivka and Savero (1970).

Corollary 4.1

For $\varepsilon > 0$ and for any $\eta,\ 0 < \eta \le 1$, $EN_\infty^\eta < \infty$, if $\sum_{k=1}^{\infty} n_k^{\eta-1} P\left(M_{N_{n_k}} \le (\theta \pm \varepsilon)\beta\left(N_{n_k}\right)\right) < \infty$.

Proof

First we show that for $\eta = 1$, $EN_\infty < \infty$ and then claim that, the existence of lower moments follows from that of the higher
moments.

Observe that

$$EN_\infty(\varepsilon) = \sum_{k\ge 1} P\left(M_{N_{n_k}} \le (\theta \pm \varepsilon)\ \beta\left(N_{n_k}\right)\right).$$

Following similar arguments of the proofs of (1) and (21), we can find some constant $C_1 > 0$ and $k_1 > 0$ such that, for all $k \ge k_1$,

$EN_\infty(\varepsilon) \le C_1 \sum_{k\ge 1} \frac{1}{(\log n_k)^{\left(\theta \pm \frac{\varepsilon}{2}\right)}}$ and by the definition of θ yields, $EN_\infty(\varepsilon) < \infty$ for $\eta = 1$.

Consequently, we have $EN_\infty^\eta < \infty$ for $0 < \eta \le 1$. Thus the proof of the Corollary is completed.

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