Research Paper E-ISSN NO : 2455-295X | VOLUME : 3 | ISSUE : 4 | APRIL 2017

EVALUATION OF DETERMINANT BY MATRIX ORDER CONDENSATION

Feng Cheng Chang [1]

[1] Allwave Corporation, 3860 Del Amo Blvd, Suite 404,Torrance, California.

ABSTRACT

A fast iterative method is presented for computing the determinant of any square matrix by applying the succession an algorithm of matrix order condensation. The process is very simple and straightforward. It is found that the total number of multiplication/division operations needed to compute the determinant of a square matrix is less than 2/3 of that required for the product of two square matrices of an identical size.

Keywords: *Determinant, Matrix Inversion, Matrix Multiplication, Recursive Algorithm, Matrix Order Condensation, Matrix Order Expansion.*

Introduction

An iterative algorithm of matrix order condensation is developed for computing the determinant of any general square matrix. The process is very simple and straightforward and involves only simple matrix oprations.

Computer listings in MATLAB are provided, and typical numerical examples are given to show the merit of the approach presented.

Formulation

The determinant of any given square matrix [*M*] can be evaluated as follows by an iterative algorithm relying upon matrix order condensation.

At the beginning of the iterative process, *k* = 0, the given square matrix [*M*] of order *N* x *N* is denoted as [*M*] = [M_0]. Then in the *k*-th step of the following process, *k* = 1, 2, ... , *K*, the matrix [M_{k-1}] of order $n_{k-1} \times n_{k-1}$ is given by

$$\left[M_{k-1}\right] = \begin{bmatrix} P_k & u_k \\ v_k & W_k \end{bmatrix},$$

which contains four sub-matrices, P_k , u_k , v_k , and W_k, of order $m_k \times m_k$, $m_k \times n_k$, $n_k \times m_k$, and $n_k \times n_k$, respectively, where $n_{k-1} = n_k + m_k$, $n_{k-1} = m_k + m_{k+1} + \cdots + m_K$, and $n_0 = N$, $n_{K-1} = m_K$, $n_K = 0$.

The condensed matrix [M_k] is thus directly computed from these four matrices, provided that the pivot matrix is P_k is not singular,

$$\left[M_k\right] = \left[W_k - v_k P_k^{-1} u_k\right].$$

Then we have

$$\det\left[M_{k-1}\right] = \det\left[P_k\right] \cdot \det\left[M_k\right].$$

The determinant of this given matrix [*M*] = [M_0] is therefore determined after performing a total of *K* process steps:

$$\det\left[M\right] = \det\left[P_1\right] \cdot \det\left[P_2\right] \cdot \cdots \cdot \det\left[P_K\right].$$

Proof ***The proof of the algorithm is quite simple and straightforward. Since***

$$\left[M_{k-1}\right] = \begin{bmatrix} P_k & u_k \\ v_k & W_k \end{bmatrix} = \begin{bmatrix} I_k & \\ v_k P_k^{-1} & I_{k+1} \end{bmatrix} \begin{bmatrix} P_k & \\ & W_k - v_k P_k^{-1} u_k \end{bmatrix} \begin{bmatrix} I_k & P_k^{-1} u_k \\ & I_{k+1} \end{bmatrix},$$

it follows that

$$\det\left[M_{k-1}\right] = \det \begin{bmatrix} P_k & u_k \\ v_k & W_k \end{bmatrix} = \det \begin{bmatrix} P_k & \\ & M_k \end{bmatrix} = \det\left[P_k\right] \cdot \det\left[M_k\right].$$

This completes the proof of the algorithm.

The algorithm can be further modified so that all sub-matrices are not necessarily solid matrices. The pivot matrix P_k, which is formed from the selected rows and columns manually in every iteration step, need not to be located along the diagonal. In fact the modified algorithm is equivalent to the original algorithm with the given matrix after the rows and columns are rearranged. The desired results are then obtained from the computed results after restoring both rows and columns into their original orders.

Computer routine

Two computer programs, derived from the algorithm and its modification, are presented as MATLAB routines.

(1)

```
function detM = det_m(M)

%   Finding determinant of a square matrix

%   by Condensation --- along diagonal.


          Z = [ ];

       for k = 1:length(M),

          nm = length(M); M,

          m = input('Pivot size = ');

        if m > nm, m = nm;   end;

          n = nm-m;

          P = M([1:m],[1:m]);

          V = M([m+1:nm],[1:m]);

          U = M([1:m],[m+1:nm]);

          W = M([m+1:nm],[m+1:nm]);

          nM = [P,U;V,W];

          iP = inv(P);

          dP = det(P);

           k,n,m,nM,iP,dP,   disp(' ');

          Z = [Z,dP];

        if nm <= m,   break, end;

          M = W-V*iP*U;

       end;

          detM = prod(Z); Z,
```

(2)

```
function detM = det_pq(M)

%   Finding determinant of a square matrix

%   by Condensation --- select rows & cols.


          Z = [ ];
```

```
for k = 1:length(M),
     nm = length(M); M,
     pq = input('[rows;cols] =   ');
     p = sort(pq(1,:));
     q = sort(pq(2,:));
     m = length(p);
     n = nm-m;
     e = (-1)^sum(pq(:));
     W = M(setdiff(1:nm,p),setdiff(1:nm,q));
     V = M(setdiff(1:nm,p),q);
     U = M(p,setdiff(1:nm,q));
     P = M(p,q);
     nM = [P,U;V,W];
     iP = inv(P);
     dP = det(P);
      k,n,m,nM,iP,   disp(' ');
     Z = [Z,dP*e];
  if nm <= m,   break, end;
     M = W-V*iP*U;
end;
     detM = prod(Z); Z,
```

Examples

For a given matrix [*M*] of order 6x6,

$$[M]=\begin{bmatrix} -1 & 5 & 8 & 3 & -4 & 3 \\ -2 & -5 & 4 & 3 & 0 & -1 \\ -3 & -2 & 5 & 0 & 7 & 4 \\ 2 & -4 & 1 & 2 & 0 & 5 \\ -6 & -2 & 4 & -7 & -1 & 6 \\ -2 & 7 & -9 & 1 & 3 & -2 \end{bmatrix}$$

its determinant det $[M]$ may be obtained in 4 different schemes by applying the derived MATLAB routines:

(1) Run detM = det_m(M), with inputs: m = 1, 1, 1, 1, 1, 1.

$k = 1, \quad m_1 = 1, \quad n_1 = 5$

$$[M_0] = [M] = \begin{bmatrix} P_1 & u_1 \\ v_1 & W_1 \end{bmatrix} = \left[\begin{array}{r|rrrrr} -1 & 5 & 8 & 3 & -4 & 3 \\ \hline -2 & -5 & 4 & 3 & 0 & -1 \\ -3 & -2 & 5 & 0 & 7 & 4 \\ 2 & -4 & 1 & 2 & 0 & 5 \\ -6 & -2 & 4 & -7 & -1 & 6 \\ -2 & 7 & -9 & 1 & 3 & -2 \end{array}\right]$$

$k = 2, \quad m_2 = 1, \quad n_2 = 4$

$$[M_1] = \left[W_1 - v_1 P_1^{-1} u_1\right] = \begin{bmatrix} P_2 & u_2 \\ v_2 & W_2 \end{bmatrix} = \left[\begin{array}{r|rrrr} -15 & -12 & -3 & 8 & -7 \\ \hline -17 & -19 & -9 & 19 & -5 \\ 6 & 17 & 8 & -8 & 11 \\ -32 & -44 & -25 & 23 & -12 \\ -3 & -25 & -5 & 11 & -8 \end{array}\right]$$

$k = 3, \quad m_3 = 1, \quad n_3 = 3$

$$[M_2] = \left[W_2 - v_2 P_2^{-1} u_2\right] = \begin{bmatrix} P_3 & u_3 \\ v_3 & W_3 \end{bmatrix} = \left[\begin{array}{r|rrr} -5.4 & -5.6 & 9.9333 & 2.9333 \\ \hline 12.2 & 6.8 & -4.8 & 8.2 \\ -18.4 & -18.6 & 5.9333 & 2.9333 \\ -22.6 & -4.4 & 9.4 & -6.6 \end{array}\right]$$

$k = 4, \quad m_4 = 1, \quad n_4 = 2$

$$[M_3] = \left[W_3 - v_3 P_3^{-1} u_3\right] = \begin{bmatrix} P_4 & u_4 \\ v_4 & W_4 \end{bmatrix} = \left[\begin{array}{r|rr} -5.8519 & 17.6420 & 14.8272 \\ \hline 0.4815 & -27.9136 & -7.0617 \\ 19.0370 & -32.1728 & -18.8765 \end{array}\right]$$

$k = 5, \quad m_5 = 1, \quad n_5 = 1$

$$[M_4] = \left[W_4 - v_4 P_4^{-1} u_4\right] = \begin{bmatrix} P_5 & u_5 \\ v_5 & W_5 \end{bmatrix} = \left[\begin{array}{r|r} -26.4620 & -5.8418 \\ \hline 25.2194 & 29.3586 \end{array}\right]$$

$k = 6, \quad m_6 = 1, \quad n_6 = 0$

$$[M_5] = \left[W_5 - v_5 P_5^{-1} u_5\right] = [P_6] = [23.7912]$$

$$[M_6] = [\quad]$$

Then

$$\det[M] = P_1 \cdot P_2 \cdot P_3 \cdot P_4 \cdot P_5 \cdot P_6 = -298410.$$

(2) Run detM = det_m(M), with inputs: m = 3, 3

$k = 1, \quad m_1 = 3, \quad n_1 = 3$

$$[M_0] = [M] = \begin{bmatrix} P_1 & u_1 \\ v_1 & W_1 \end{bmatrix} = \left[\begin{array}{rrr|rrr} -1 & 5 & 8 & 3 & -4 & 3 \\ -2 & -5 & 4 & 3 & 0 & -1 \\ -3 & -2 & 5 & 0 & 7 & 4 \\ \hline 2 & -4 & 1 & 2 & 0 & 5 \\ -6 & -2 & 4 & -7 & -1 & 6 \\ -2 & 7 & -9 & 1 & 3 & -2 \end{array}\right]$$

$k = 2, \quad m_2 = 3, \quad n_2 = 0$

$$[M_1]=[W_1-v_1P_1^{-1}u_1]=[P_2]=\begin{bmatrix} -5.8519 & 17.6420 & 14.8272 \\ 0.4815 & -27.9136 & -7.0617 \\ 19.0370 & -32.1728 & -18.8765 \end{bmatrix}$$

$$[M_2] \;=\; [\;\;]$$

Then

$$\det[M] \;=\; \det[P_1]\cdot\det[P_2] = \; -298410.$$

(3) Run detM = det_pq(M), inputs: [rows; cols] = [4;3], [2;3], [3;3], [3;1],[2;1],[1;1].

$$[M_0] = \;[M] = \begin{bmatrix} -1 & 5 & 8 & 3 & -4 & 3 \\ -2 & -5 & 4 & 3 & 0 & -1 \\ -3 & -2 & 5 & 0 & 7 & 4 \\ 2 & -4 & 1 & 2 & 0 & 5 \\ -6 & -2 & 4 & -7 & -1 & 6 \\ -2 & 7 & -9 & 1 & 3 & -2 \end{bmatrix}$$

$k = 1$, $[r_1;\ c_1]=[4;\ 3]$

$$\begin{bmatrix} P_1 & u_1 \\ v_1 & W_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -4 & 2 & 0 & 5 \\ 8 & -1 & 5 & 3 & -4 & 3 \\ 4 & -2 & 5 & 3 & 0 & -1 \\ 5 & -3 & -2 & 0 & 7 & 4 \\ 4 & -6 & -2 & -7 & -1 & 6 \\ -9 & -2 & 7 & 1 & 3 & -2 \end{bmatrix}$$

$$z_1 = \;(-1)^{r_1+c_1}\det[P_1] = \; -1.$$

$$[M_1]=[W_1-v_1P^{-1}{}_1u_1]=\begin{bmatrix} -17 & 37 & -13 & -4 & -37 \\ -10 & 11 & -5 & 0 & -21 \\ -13 & 18 & -10 & 7 & -21 \\ -14 & 14 & -15 & -1 & -14 \\ 16 & -29 & 19 & 3 & 43 \end{bmatrix}$$

$k = 2$, $[r_2;\ c_2]=[2;\ 3]$

$$\begin{bmatrix} P_2 & u_2 \\ v_2 & W_2 \end{bmatrix} = \begin{bmatrix} -5 & -10 & 11 & 0 & -21 \\ -13 & -17 & 37 & -4 & -37 \\ -10 & -13 & 18 & 7 & -21 \\ -15 & -14 & 14 & -1 & -14 \\ 19 & 16 & -29 & 3 & 43 \end{bmatrix}$$

$$z_2 = \;(-1)^{r_2+c_2}\det[P_2] = \; +5.$$

$$\left[M_2\right]=\left[W_2-v_2P_2^{-1}u_2\right]=\begin{bmatrix} 9 & 8.4 & -4 & 17.6 \\ 7 & -4 & 7 & 21 \\ 16 & -19 & -1 & 49 \\ -22 & 12.8 & 3 & -36.8 \end{bmatrix}$$

$k = 3,\qquad [\mathrm{r}_3;\ \mathrm{c}_3]=[3;\ 3]$

$$\begin{bmatrix} P_3 & u_3 \\ v_3 & W_3 \end{bmatrix}=\left[\begin{array}{c|ccc} -1 & 16 & -19 & 49 \\ \hline -4 & 9 & 8.4 & 17.6 \\ 7 & 7 & -4 & 21 \\ 3 & -22 & 12.8 & -36.8 \end{array}\right]$$

$$z_3 = (-1)^{r_3+c_3}\det\left[P_3\right]= -1.$$

$$\left[M_3\right]=\left[W_3-v_3P_3^{-1}u_3\right]=\begin{bmatrix} -55 & 84.4 & -178.4 \\ 119 & -137 & 364 \\ 26 & -44.2 & 110.2 \end{bmatrix}$$

$k = 4,\qquad [\mathrm{r}_4;\ \mathrm{c}_4]=[3;\ 1]$

$$\begin{bmatrix} P_4 & u_4 \\ v_4 & W_4 \end{bmatrix}=\left[\begin{array}{c|cc} 26 & -44.2 & 110.2 \\ \hline -55 & 84.4 & -178.4 \\ 119 & -137 & 364 \end{array}\right]$$

$$z_4 = (-1)^{r_4+c_4}\det\left[P_4\right]= +26.$$

$$\left[M_4\right]=\left[W_4-v_4P_4^{-1}u_4\right]=\begin{bmatrix} -9.1 & 54.7154 \\ 65.3 & -140.3769 \end{bmatrix}$$

$k = 5,\qquad [\mathrm{r}_5;\ \mathrm{c}_5]=[2;\ 1]$

$$\begin{bmatrix} P_5 & u_5 \\ v_5 & W_5 \end{bmatrix}=\left[\begin{array}{c|c} 65.3 & -140.3769 \\ \hline -9.1 & 54.7145 \end{array}\right]$$

$$z_5 = (-1)^{r_5+c_5}\det\left[P_5\right]= -65.3.$$

$$\left[M_5\right]=\left[W_5-v_5P_5^{-1}u_5\right]=\left[35.153\right]$$

$k = 6,\qquad [\mathrm{r}_6;\ c_6]=[1;\ 1]$

$$\left[P_6\right]= \left[35.135\right]$$

$$z_6 = (-1)^{r_6+c_6}\det\left[P_6\right]= +35.135.$$

$$\left[M_6\right]= [\quad]$$

Then,

$$\det[M] = z_1 \cdot z_2 \cdot z_3 \cdot z_4 \cdot z_5 \cdot z_6 = -298410.$$

(4) Run detM = det_pq(M), with inputs: [rows; cols] = [2 4 5; 1 4 5], [1; 2], [1 2; 1 2].

$$[M_0] = [M] = \begin{bmatrix} -1 & 5 & 8 & 3 & -4 & 3 \\ -2 & -5 & 4 & 3 & 0 & -1 \\ -3 & -2 & 5 & 0 & 7 & 4 \\ 2 & -4 & 1 & 2 & 0 & 5 \\ -6 & -2 & 4 & -7 & -1 & 6 \\ -2 & 7 & -9 & 1 & 3 & -2 \end{bmatrix}$$

$k = 1$, $[r_1;\ c_1] = [2\,4\,5;\ 1\,4\,5]$

$$\begin{bmatrix} P_1 & u_1 \\ v_1 & W_1 \end{bmatrix} = \left[\begin{array}{ccc|ccc} -2 & 3 & 0 & -1 & 3 & -4 \\ 2 & 2 & 0 & -3 & 0 & 7 \\ -6 & -7 & -1 & -2 & 1 & 3 \\ \hline -5 & 4 & -1 & 5 & 8 & 3 \\ -4 & 1 & 5 & -2 & 5 & 4 \\ -2 & 4 & 6 & 7 & -9 & -2 \end{array}\right]$$

$$z_1 = (-1)^{\Sigma r_1 + c_1} \det[P_1] = -10.0$$

$$[M_1] = [W_1 - v_1 P_1^{-1} u_1] = \begin{bmatrix} 73.4 & -27.5 & -84.9 \\ -113.2 & 59.5 & 161.7 \\ -39.0 & 13.0 & 66.0 \end{bmatrix}$$

$k = 2$, $[r_2;\ c_2] = [1;\ 2]$

$$\begin{bmatrix} P_2 & u_2 \\ v_2 & W_2 \end{bmatrix} = \left[\begin{array}{c|cc} -27.5 & 73.4 & -84.9 \\ \hline 59.5 & -113.2 & 161.7 \\ 13.0 & -39.0 & 66.0 \end{array}\right]$$

$$z_2 = (-1)^{\Sigma r_2 + c_2} \det[P_2] = +27.5$$

$$[M_2] = [W_2 - v_2 P_2^{-1} u_2] = \begin{bmatrix} 45.6109 & -21.9927 \\ -4.3018 & 25.8655 \end{bmatrix}$$

$k = 3$, $[r_3;\ c_3] = [1\,2;\ 1\,2]$

$$[P_3] = \begin{bmatrix} 45.6109 & -21.9927 \\ -4.3018 & 25.8655 \end{bmatrix}$$

$$z_3 = (-1)^{\Sigma r_3 + c_3} \det[P_3] = +1085.1$$

$$[M_3] = [\quad]$$

Then

$$\det[M] = z_1 \cdot z_2 \cdot z_3 = -298410.$$

Conclusion

A simple method has been developed for finding the determinant of a given square matrix of high order. The process involves successive applications of an algorithm for matrix order condensation. The algorithm may be modified to resolve the problem in the event that the pivot matrix becomes singular during any iteration step. The sign of the computed result must be adjusted regarding to the swapping of the rows and columns of the pivot matrix at each process step.

When compared to various approaches available in the literature [1]-[4], the process presented is very compact, efficient, and involves only the simple elementary arithmetical operations of addition, subtraction, multiplication, and division. It is shown that, no matter what size of pivot matrix chosen at each step, the total number of multiplication/division operations needed to compute the determinant of a given N x N matrix is found to be $\frac{2}{3}N^3 - N^2 + \frac{4}{3}N - 1$, where N^3 is the total number of multiplication operations required to compute the product of any two N x N matrices.

If we are interested in both determinant and inverse of any given square matrices, the algorithm of matrix order expansion as found in [7] may be useful.

Acknowledgements

The author would like to acknowledge the useful help from Dr. George Cheng, Dr. Jan Grzesik, Dr. Young Chu, Ms. Lala Xu and Mr. Felix Wong of Allwave Corporation.

REFERENCES

1. C.T. Su and F.C. Chang, "Quick evaluation of determinant," Appl. Math. & Compu. 75, 1996, p.117-118.

2. F.C. Chang, "Determinant of matrix by order condensation," British J. of Math. & Comput. Science, 4(13), 2014, pp.1843-1848.

3. O. Rezaifar and H. Rezaee, "Anew approach for finding the determinant of matrices," Appl. Math. & Compu. 188, 2007, pp.1445-1454.

4. T. Sogabe, "On a two-term recurrence for the determinant of a general matrix," Appl. Math. & Compu. 187, 2007, pp.758-788.

5. A. R. Moghaddamfar, S. Navid Salehy and S. Nima Salehy, "The determinant of matrices with recursive entries," Linear Algeb.& Its appl., 428, 2008, pp.2468-2481.

6. R.S. Bird,"A simple division-free algorithm for computing determinants," Information Processing Letters, 111, 2011, pp.1072-1074.

7. F.C.Chang,"Inverse and determinant of a matrix by order expansion and condensation," IEEE Antenas and Propagation Magazine, 57(1), 2015, pp.28-32.